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The singular coagulation equation with multiple fragmentation

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Abstract. In this paper, we prove the global existence and uniqueness of the solutions to the initial-value problem for the coagulation–fragmentation equation with singular coagulation kernel and multiple fragmentation kernel. The solution obtained in this case also satisfies the mass conservation law. The proof is based on strong convergence methods applied to suitably chosen unbounded coagulation kernels having singularities in both the coordinate axes and satisfying certain growth conditions, which can possibly reach up to a quadratic growth at infinity, and the fragmentation kernel covers a very large class of unbounded functions.

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1. Introduction

The process by which two or more particles undergo changes in its physical properties is called the particulate process. Before we proceed to explain our problem, let us first give an impression on the significant applications of the particulate processes in real life. Particulate processes are well known in various branches of engineering including crystallization, precipitation, polymerization, and various particle-related engineering problems. Its applications can be found in many areas including chemistry (reacting polymers), physics (aggregation of colloidal particles, growth of gas bubbles in solids), astrophysics (formation of stars and planets), and meteorology (merging of drops in atmospheric clouds). In drying, particulate process is used in two ways. One application concerns with the drying of particles in a continuous fluidized bed dryer, while other involves the process of simultaneous particle size enlargement and drying. The spontaneous collisions of particles with other particles result in the change in its mass, shape, size, volume, etc. The change in the particle number density $f(x,t) \geq 0$, for particles of volume $x \geq 0$ at some time $t \geq 0$ in a physical system undergoing coagulation and fragmentation process is described by the following equation, popularly known as the population balance equation (PBE),

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,\mathrm{d}y - f(x,t)\int_0^\infty K(x,y)f(y,t)\,\mathrm{d}y + \int_x^\infty b(x,y)S(y)f(y,t)\,\mathrm{d}y - S(x)f(x,t) \tag{1}$$

with the initial data,

$$f(x,0) = f_0(x) \geq 0. \tag{2}$$

In PBE (1), $K(x, y)$ denotes the coagulation kernel of the system. It describes the rate at which particles of size x unite with particles of size y to form particles of size $x + y$. The fragmentation kernels are defined as follows:

- $b(x, y)$ is the breakage function denoting the probability density function for the formation of particles of size x from the particles of size y. It is nonzero only for $x < y$.
- The selection function $S(y)$ describes the rate at which particles of size y are selected to break.

The breakage function $b(x, y)$ satisfies the properties,

$$\int_0^y x b(x, y)\,\mathrm{d}x = y, \ \forall\, y > 0 \tag{3}$$

and,

$$\int_0^y b(x, y)\,\mathrm{d}x = \nu(y) \le N, \ \forall\, y > 0 \ \text{ and } \ b(x, y) = 0, \ \forall\, x \ge y. \tag{4}$$

For the total mass in the system to remain conserved during fragmentation events, $b(x, y)$ is considered to satisfy the Eq. (3). It represents that when a particle of mass y breaks into smaller fragments, then the total mass of the fragments formed is equal to y. In Eq. (4), $\nu(y)$ denotes the total number of fragments of particles produced due to the breakage of the particle of size y, which is assumed to be bounded by a constant number N. From the physical point of view, it is clear that $K(x, y)$ must be nonnegative and symmetric, *i.e.*, $K(x, y) = K(y, x), \ \forall\ 0 \le x, y < \infty$. Some authors use $\Gamma(y, x)$ as the notation for multiple fragmentation kernel, it denotes the rate at which the particles of size y are breaking into smaller fragments x. The relationships between $\Gamma(y, x)$, $S(x)$, and $b(x, y)$ are defined as follows:

$$S(x) = \int_0^x \frac{y}{x}\Gamma(x, y)\,\mathrm{d}y, \ \ b(x, y) = \frac{\Gamma(y, x)}{S(y)}.$$

In Eq. (1),

1. the first term describes the formation of particles of size x, by the coagulation of particles of size $x - y$ with y;
2. the second term describes the loss of particles of size x, because of the coagulation of particle of size y with x forming the particle of size $x + y$;
3. the third term gives the formation of particle of size x, due to the breakage of particles $y(> x)$ into size x; and finally,
4. the last term implies the loss of particle of size x, because of its breakage into smaller fragments.

The first and the third integrals in (1) describe the formation of particle of size x in the system, hence they are called the birth terms, while the second and the fourth integrals describe the loss of particles of size x from the system and so are called death terms.

In the study of any equation, one of the first mathematical questions is Does the solution of the equation exist? If it exists, then whether it is unique or not? There are many results on the existence and uniqueness of solutions to the various forms of the coagulation–fragmentation equation, which have been obtained by applying different methods for different kernels. A precise review through the existing literature gives us an idea on the conditions that are required to show the well-posedness of the coagulation–fragmentation equation. These conditions include some bounds on the kernels as well as the finiteness of the total number of particles $\left(\int_0^\infty f_0(x)\,\mathrm{d}x\right)$ and total mass of the particles $\left(\int_0^\infty x f_0(x)\,\mathrm{d}x\right)$ taken initially.

Melzak [18] and McLeod [17] were the first to discuss the existence and uniqueness of solutions for the PBEs. Later, Ball and Carr [2] have studied the discrete system of equations having mass conservative solutions and Stewart [19,21,22] as well as Laurençot [13,14] treated the continuous equations using compactness methods in the space of integrable functions. The main difference between the discrete and the continuous models is that the space l^1 is contained in the space l^∞ for the discrete case, whereas for the continuous case, the space L^∞ is contained in L^1. For this reason, while dealing with the continuous version of the PBEs, it includes some additional estimates. Laurençot and Mischler [14] had discussed the relationship between the discrete and the continuous models of the PBEs. In the first study of the PBEs, Melzak [18] had established the existence and uniqueness result for coagulation fragmentation equations with multiple fragmentation having bounded kernels. McLaughlin et al. [16] established the existence and uniqueness of solutions to the multiple fragmentation equation under the condition that $S(x) \leq C_n < \infty$, for all $x \in]0, n]$, $n > 0$, where the sequence C_n may be unbounded. This result was extended by McLaughlin et al. [15] to the combined equation for coagulation and multiple fragmentation under the assumptions that $K(x, y)$ is constant and $\Gamma \in L^1(]0, \infty[\times]0, \infty[)$. Using similar arguments, Lamb [12] discussed the existence of solutions to (1), (2) under the less restrictive conditions that $K(x, y)$ is bounded, $S(x)$ satisfies a linear growth condition, and $b(x, y)$ is such that the breakup of a particle of size y is a mass-conserving process that produces a finite number of smaller particles independent of y. Dobovskiĭ and Stewart [6,7] had discussed on the existence of solutions for the PBEs where the coagulation kernel satisfies a possible linear growth at infinity and the binary fragmentation kernels covers a huge class of unbounded functions. The work of Stewart [20,22] was extended by Giri et al. [11] for a larger class of coagulation kernel and multiple fragmentation kernels.

All the results mentioned above are valid for different forms of PBEs with nonsingular coagulation kernels. To our knowledge, the literature has lot of experimental works over the different forms of PBEs with singular coagulation kernels. Smoluchowski was the first to consider the pure aggregation PBE with singular coagulation kernel showing Brownian motion of the particles. Later, many authors had their experimental results validated with different forms of the kernels viz. shear motion of gas via the shear kernels, kinetics of granulation of particles through the granulation kernels, equipartition of kinetic energy of the particles using the EKE kernels, etc. and in most of these cases, the singular coagulation kernels have been used. Many of these singular coagulation kernels have been formulated largely upon experimental observations but they have no theoretic foundations.

The very recent work concerning the existence and uniqueness theory of solutions for the PBE with singular coagulation kernel and multiple fragmentation kernel was done by Camejo [3,4]. In [3], Camejo has considered the bound him over the coagulation kernel to be $K(x, y) \leq k\frac{(1+x)^{\lambda}(1+y)^{\lambda}}{(xy)^{\sigma}}$, where $k(> 0)$ is a constant, $\sigma \in \left[0, \frac{1}{2}\right]$ and $\lambda - \sigma \in [0, 1[$. He followed the result, obtained by Stewart [22], for the PBE (1) with the coagulation kernels having singularity in the axes $x = 0, y = 0$. The proof for the existence and uniqueness of the solutions is based upon the weak L^1 compactness methods applied to suitably chosen approximating equations in the space $L^1(]0, \infty[\times[0, T])$. For the fragmentation kernel, the bound over the selection function is considered to be $S(x) \leq x^{\theta}$, where $\theta \in [0, 1[$. Moreover, while studying the case of existence theory, he had to impose a number of assumptions over the breakage function. In the uniqueness of the solutions, the restriction $\lambda - \sigma \in \left[0, \frac{1}{2}\right]$ had already limited the result of Camejo [3] to a subset of the kernels of the class as defined in the existence theory. Due to this restriction, the theory is unable to show the uniqueness of solutions for the PBEs with equipartition of kinetic energy kernels. Further, the condition $\theta \leq \lambda - \sigma$ had shrunk the uniqueness result for a much more restricted class of fragmentation kernels.

In this present article, we have developed the existence and uniqueness theory for the mass-conserving solution of (1), (2). The theory is motivated upon the strong convergence criterion of Dobovskiĭ [7] for the continuous functions applied to a suitably chosen approximating equations. Here, the coagulation term satisfies $K(x, y) \leq k\frac{(1+x)^{\lambda}(1+y)^{\lambda}}{(xy)^{\sigma}}$, $k(> 0)$ is a constant, $\sigma \in \left[0, \frac{1}{2}\right]$ and $\lambda - \sigma \in [0, 1]$. So, the coagulation kernel has singularity on both the coordinate axes, and moreover, it includes a larger class of functions compared to the kernel used in [3]. With this bound, we are able to include the Smoluchowski's kernel of Brownian motion, the equipartition of kinetic energy (EKE) kernel, the granulation kernels, the shear kernels (both the linear and the nonlinear velocity profiles), activated sludge flocculation kernel by Ding et al. and the kernel showing aerosol dynamics by Friedlander into our consideration. These kernels are very important because of their immense practical use. Moreover, when $\lambda - \sigma = 1$, then we will get the existence, uniqueness result for the product kernels. The novelty of our coagulation kernel is that, besides having singularities over the axes, it also satisfies certain growth conditions that can possibly be the quadratic growth at infinity.

In the fragmentation kernel, we have used the bound over the selection function as $S(x) \leq S_1 x^{\beta}$, where $S_1(>0)$ is a constant and $0 < \beta$ with $x \in [0, \infty[$. Due to the above-mentioned bounds of the coagulation kernels, it is possible for the PBE (1) that tends to loose mass at some point of time. Therefore, a *strong fragmentation criterion* has been implemented over the selection function to refrain the system from attaining gelation. So, for this reason, following [9], we put the bound $S_0 x^{\alpha} < S(x)$, where $S_0(>0)$ is a constant and $\alpha > 0$ with $x \geq x_0 \geq 1$ over the selection function to control the rate of formation of the very large size of particles that tend to move out from the system resulting to the mass loss in the system. With these bounds, we are able to develop the existence theory over a large class of unbounded selection functions. A lesser number of restrictions have been put over the breakage function as compared to [3]. To study the uniqueness result of the solution, we had to assume an additional restriction over the selection function. Despite that restriction, our theory also ensures the uniqueness of the solutions for all the kernels mentioned above in the existence part, which are very important for their practical usage purpose.

The outline of this paper is as follows. Next, Sect. 2 presents the existence theory of the solutions. The Sect. 3 presents the mass conservation law satisfied by the density function. The uniqueness results have been proved in Sect. 4. Finally, conclusions are made in the last Sect. 5.

2. Existence theorem

We now introduce the functional spaces,

$$\Pi = \{(x,t) : 0 < x < \infty, 0 \leq t \leq T\},$$

where $T(>0)$ is fixed and we define the space $\Pi(X_1, X_2, T)$ be the rectangle,

$$\Pi(X_1, X_2, T) = \{(x,t) : X_1 \leq x \leq X_2, 0 \leq t \leq T\},$$

where $0 < X_1 < X_2$ are finite numbers.

Let us define the spaces,

$$\Omega_{r_1,r_2}(T) = \left\{ \text{all continuous functions } f(x,t) : \sup_{0 \leq t \leq T} \int_0^{\infty} \left(x^{r_1} + \frac{1}{x^{r_2}} \right) |f(x,t)| \, \mathrm{d}x = \|f\|_{r_1,r_2} < \infty \right\},$$

where $r_1 \geq 1$ and $0 < r_2 < 1$. The cone of the nonnegative functions in $\Omega_{r_1,r_2}(T)$ is denoted as $\Omega^{+}_{r_1,r_2}(T)$. Now, in this section to show the existence result, we need to prove the following theorem.

Theorem 2.1. *Let the functions $K(x,y)$, $b(x,y)$ be continuous, nonnegative over $]0,\infty[\times]0,\infty[$, $S(x)$ be continuous and nonnegative over $]0,\infty[$, and $K(x,y)$ is symmetric $\forall\, x,y \in]0,\infty[$. Suppose*

(i) $\forall\, x,y \in]0,\infty[,\ K(x,y) \leq k\frac{(1+x)^{\lambda}(1+y)^{\lambda}}{(xy)^{\sigma}}$ *with $k > 0$ a constant, $\sigma \in \left[0, \frac{1}{2}\right]$ and $\lambda - \sigma \in [0,1]$,*

(ii) $S(x) \leq S_1 x^{\beta}$, $\forall\, x \in]0,\infty[$ *where $S_1(>0)$ a constant and $0 < \beta \leq r_1 - 1$,*

(iii) $S_0 x^{\alpha} < S(x)$, *where $S_0(>0)$ a constant, $0 < \alpha \leq r_1 - 1$ and $x \geq x_0$, where $x_0(\geq 1)$ is a large number,*

(iv) *for some real number γ such that $0 < \gamma < 1$, $\int_0^y \frac{1}{x^{\gamma}} b(x,y)\, dx \leq \frac{N_0}{y^{\gamma}}$ where $N_0 > 0$ is a constant,*

(v) $\lim_{y\to\infty} \sup_{x\in[x_1,x_2]} b(x,y) \leq \bar{b}$, $\forall\ 0 < x_1 < x_2 < \infty$ *and $\bar{b}$ is a constant,*

and let the initial data satisfy $f_0(x) \in \Omega^{+}_{r_1,r_2}(0)$, then the problem (1), (2) has at least one solution in $\Omega^{+}_{r_1,r_2}(T)$.

Remark 2.1. The condition (iv) in Theorem 2.1 covers a very large class of breakage functions. Not that any breakage function satisfying mass conservation property (3) can be written in the form $b(x,y) = \sum_i C_i\xi_i(x,y)$, where $\xi_i(x,y) = C\frac{x^{\eta_i-1}}{y^{\eta_i}}$, where $\eta_i > 0$ and C, C_i are suitably chosen constants. But one can easily show that any such breakage function satisfies the condition (iv) of Theorem 2.1.

Remark 2.2. The condition (iii) in Theorem 2.1 ensures that the fragmentation rate is sufficiently strong with respect to the coagulation rate. In general, when $\lambda-\sigma \in]\frac{1}{2},1]$, the selection rate must have to satisfy $S(x) > S_0x^{\theta}$, where $\theta > 2(\lambda-\sigma)-2$, $S_0(>0)$ a constant and $x \geq 1$. So, for $\lambda-\sigma \in]\frac{1}{2},1]$, with the above criterion and $\theta > -1$, a control over the second and the higher moments of the solution $f(x,t)$ for positive times can be established by following the work [9].

Proof. Let $K(x,y)$ and $S(x)$ satisfy the conditions of Theorem 2.1 for all $x, y \in]0,\infty[$.

Following the "kernel-truncation" idea of Dobovskiǐ [7], and Camejo [3], we construct the sequence of continuous kernels $\{K_n,S_n\}_{n=1}^{\infty}$ from the class of kernels as defined in Theorem 2.1, with compact support for each $n \geq 1$ as follows

$$K_n(x,y)\begin{cases} = K(x,y), & \text{when} \frac{1}{n} \leq x,y \leq n,\\ \leq K(x,y), & \text{elsewhere,}\end{cases}$$

and

$$S_n(x)\begin{cases} = S(x), & \text{when} \frac{1}{n} \leq x \leq n,\\ \leq S(x), & \text{elsewhere.}\end{cases}$$

These modified kernels $K_n(x,y)$ and $S_n(x)$ are so constructed that they are continuous functions of x, y and they continuously decreases to "zero" outside the intervals $\left[\frac{1}{n},n\right] \times \left[\frac{1}{n},n\right]$ and $\left[\frac{1}{n},n\right]$, respectively, so that the above intervals can be considered to be the compact support of the kernels $K(x,y)$ and $S(x)$. Moreover, by the above construction, if the kernels $K(x,y)$ and $S(x)$ are continuous over a closed interval, then the family of continuous kernels $\{K_n\}$ and $\{S_n\}$ is equicontinuous over that interval. Now, in accordance with the works of Dobovskiǐ [7] and Stewart [22], the sequences of functions $\{K_n,S_n\}_{n=1}^{\infty}$ generate on Π a sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative continuous solutions to the problem (1), (2) with the kernels K_n, S_n. These solutions $f_n(x,t)$ belong to the space $\Omega^{+}_{r_1,r_2}(T)$. Therefore, the PBE (1) is written as

$$\begin{aligned}\frac{\partial f_n(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K_n(x-y,y)f_n(x-y,t)f_n(y,t)\,\mathrm{d}y - f_n(x,t)\int_0^{\infty} K_n(x,y)f_n(y,t)\,\mathrm{d}y\\ &\quad+\int_x^{\infty} b(x,y)S_n(x)f_n(y,t)\,\mathrm{d}y - S_n(x)f_n(x,t).\end{aligned} \tag{5}$$

Let us denote the ith moment of the function $f_n(x,t)$ as

$$N_{i,n}(t) = \int_0^{\infty} x^i f_n(x,t)\,\mathrm{d}x,\ i \in \mathbb{N},\ n \geq 1. \tag{6}$$

By direct integration of Eq. (5) with an weight x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}N_{1,n}(t) = \int_0^{\infty} x\frac{\partial f_n(x,t)}{\partial t}\,\mathrm{d}x.$$

Due to the compact support of the kernels $K_n(x,y)$ and $S_n(x)$, we get that all the integrals obtained in the right-hand side of the above equation are finite and they cancel out. Hence, we get

$$N_{1,n}(t) = \bar{N}_1 = \text{ constant}, n \geq 1, \quad t > 0. \tag{7}$$

Now, integrating (5) with the weight x^2 and using the bounds over the kernels, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}N_{2,n}(t) &= \int_0^\infty x^2 \frac{\partial f_n(x,t)}{\partial t}\,\mathrm{d}x\\ &\leq \int_0^\infty \int_0^\infty xyK_n(x,y)f_n(x,t)f_n(y,t)\,\mathrm{d}x\,\mathrm{d}y\\ &\leq k\int_0^\infty \int_0^\infty (xy)^{1-\sigma}(1+x)^\lambda(1+y)^\lambda f_n(x,t)f_n(y,t)\,\mathrm{d}y\,\mathrm{d}x\\ &\leq k\left[\bar{N}_1 + N_{2,n}\right]^2.\end{aligned}$$

Hence,

$$N_{2,n} \leq \bar{N}_2 \text{ where } \bar{N}_2 > 0 \text{ is independent of } n,\ 0 \leq t \leq T,\ n \geq 1. \tag{8}$$

So, in a similar way, we can proceed further for $i = 3, 4, \ldots$ and can obtain the uniform boundedness of $N_{r_1,n}(t)$ where $n \geq 1$, $0 \leq t \leq T$, and r_1 is taken accordingly from the spaces $\Omega^+_{r_1,r_2}(T)$.

Let us first define $\lceil x \rceil$, $x \in \mathbb{R}$, to be the smallest integer not less than x. For the uniform boundedness of $N_{0,n}(t)$, we proceed as below

$$\begin{aligned}\frac{\mathrm{d}N_{0,n}}{\mathrm{d}t}(t) &= \int_0^\infty \frac{\partial}{\partial t} f_n(x,t)\,\mathrm{d}x\\ &= \int_0^\infty \left[\frac{1}{2}\int_0^x K_n(x-y,y)f_n(x-y,t)f_n(y,t)\,\mathrm{d}y - \int_0^\infty K_n(x,y)f_n(x,t)f_n(y,t)\,\mathrm{d}y\right.\\ &\quad \left. + \int_x^\infty b(x,y)S_n(y)f_n(y,t) - S_n(x)f_n(x,t)\right]\mathrm{d}x.\end{aligned} \tag{9}$$

Considering the first integral of (9), changing the order of integration and then substituting $x - y = x'$, $y = y'$ and again changing the order of integration, we get

$$\int_0^\infty \frac{1}{2}\int_0^x K_n(x-y,y)f_n(x-y,t)f_n(y,t)\,\mathrm{d}y\,\mathrm{d}x = \int_0^\infty \int_0^\infty \frac{1}{2}K_n(x,y)f_n(x,t)f_n(y,t)\,\mathrm{d}y\,\mathrm{d}x.$$

Thus, the first two integrals in (9) give

$$\int_0^\infty \int_0^x \frac{1}{2}K_n(x-y,y)f_n(x-y,t)f_n(y,t)\,\mathrm{d}y\,\mathrm{d}x - \int_0^\infty \int_0^\infty K_n(x,y)f_n(x,t)f_n(y,t)\,\mathrm{d}y\,\mathrm{d}x \leq 0.$$

Using (4), we get

$$\begin{aligned}\frac{\mathrm{d}N_{0,n}}{\mathrm{d}t}(t) &\leq \int_0^\infty \int_x^\infty b(x,y)S_n(y)f_n(y,t)\,\mathrm{d}y\,\mathrm{d}x - \int_0^\infty S_n(x)f_n(x,t)\,\mathrm{d}x\\ &= \int_0^\infty \int_0^y b(x,y)S_n(y)f_n(y,t)\,\mathrm{d}x\,\mathrm{d}y - \int_0^\infty S_n(x)f_n(x,t)\,\mathrm{d}x\\ &\leq NS_1\int_0^\infty y^\beta f_n(y,t)\,\mathrm{d}y\ \left[\int_0^y b(x,y)\,\mathrm{d}x \leq N = \text{ constant}\right]\\ &\leq NS_1 \cdot \bar{N}_{\lceil\beta\rceil} = \bar{N}_0 = \text{is a constant independent of} \quad n.\end{aligned} \tag{10}$$

Therefore, combining all the above relations, we obtain

$$N_{i,n}(t) \leq \bar{N}_i = \text{ constant (independent of} n\text{)if } t \in [0,T], \quad n \geq 1,\ 0 \leq i \leq r_1. \tag{11}$$

For any real number γ, satisfying $\sigma < \gamma < 1$ and the condition (iv) of Theorem 2.1. Then, in that case, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}N_{-\gamma,n} &\le \int_0^\infty\int_0^y \frac{1}{x^\gamma} b(x,y)S_n(y)f_n(y,t)\,\mathrm{d}y\,\mathrm{d}x - \int_0^\infty \frac{1}{y^\gamma}S_n(y)f_n(y,t)\,\mathrm{d}y\\ &\le S_1\left(N_0-1\right)\int_0^\infty y^{\beta-\gamma}f_n(y,t)\,\mathrm{d}y\\ &= S_1\left(N_0-1\right)N_{\beta-\gamma,n}.\end{aligned}$$

If $\beta-\gamma>0$, then $N_{\beta-\gamma}\le \bar{N}_{\lceil\beta-\gamma\rceil}$.

If $\beta-\gamma<0$, then we find $\frac{\mathrm{d}}{\mathrm{d}t}N_{\beta-\gamma,n}\le S_1\left(N_0-1\right)N_{2\beta-\gamma,n}$. If $2\beta-\gamma>0$, then we stop or else find $\frac{\mathrm{d}}{\mathrm{d}t}N_{2\beta-\gamma,n}$. This iteration continues till $z\beta-\gamma>0$ ($z>0$ an integer). Consequently, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}N_{-\gamma,n}\le \bar{N}_\gamma,\ \text{(a constant independent of } n).$$

Generalizing the above relation for any $0<j\le r_2$ by proceeding in a similar way, we can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}N_{-j,n}\le \bar{N}_j,\ \text{(a constant independent of } n). \tag{12}$$

We are now in a position to state the following lemma:

Lemma 2.1. *The sequence* $\{f_n\}_{n=1}^\infty$ *is relatively compact in the uniform convergence topology of continuous functions on each rectangle* $\Pi(X_1,X_2,T)$.

Proof. For proving the Lemma, we proceed as follows:

1. we first prove the uniform boundedness of the sequence $\{f_n\}_{n=1}^\infty$,
2. then the equicontinuity of the sequence $\{f_n\}_{n=1}^\infty$ with respect to the variable x, and finally,
3. the equicontinuity of the sequence $\{f_n\}_{n=1}^\infty$ with respect to time variable t.

Step I. Here, we prove that $\{f_n\}_{n=1}^\infty$ is uniformly bounded on $\Pi(X_1,X_2,T)$. Let $X=\max\left\{\frac{1}{X_1},X_2\right\}$ From Eq. (5), we have

$$\begin{aligned}\frac{\partial f_n(x,t)}{\partial t} &= \frac{1}{2}\int_0^x K_n(x-y,y)f_n(x-y,t)f_n(y,t)\,\mathrm{d}y - \int_0^\infty K_n(x,y)f_n(x,t)f_n(y,t)\,\mathrm{d}y\\ &\quad+\int_x^\infty S_n(y)b(x,y)f_n(y,t)\,\mathrm{d}y - S_n(x)f_n(x,t)\\ &\le \frac{1}{2}\int_0^x K_n(x-y,y)f_n(x-y,t)f_n(y,t)\,\mathrm{d}y + \int_x^\infty S_n(y)b(x,y)f_n(y,t)\,\mathrm{d}y\\ &\le \frac{k}{2}\int_0^x \frac{(1+x-y)^\lambda(1+y)^\lambda}{(x-y)^\sigma y^\sigma}f_n(x-y,t)f_n(y,t)\,\mathrm{d}y + S_1\int_x^\infty y^\beta b(x,y)f_n(y,t)\,\mathrm{d}y\\ &\le \frac{k}{2}\int_0^x \frac{(1+x)^\lambda(1+y)^\lambda}{(x-y)^\sigma y^\sigma}f_n(x-y,t)f_n(y,t)\,\mathrm{d}y + S_1\bar{b}\int_x^\infty y^\beta f_n(y,t)\,\mathrm{d}y\\ &\le \frac{k}{2}\left(1+X\right)^{2\lambda}\int_0^x \frac{1}{(x-y)^\sigma y^\sigma}f_n(x-y,t)f_n(y,t)\,\mathrm{d}y + S_1\bar{b}\int_x^\infty y^\beta f_n(y,t)\,\mathrm{d}y\\ &\le \frac{k}{2}(1+X)^{2\lambda}\int_0^x g_n(x-y,t)g_n(y,t)\,\mathrm{d}y + S_1\bar{b}\bar{N}_{\lceil\beta\rceil},\end{aligned}$$

where

$$g(y,s)=\frac{f(y,s)}{y^\sigma}.$$

Therefore, we have

$$\frac{\partial g_n(x,t)}{\partial t} \le \frac{k}{2}(1+X)^{2\lambda}X^{\sigma}\int_0^x g_n(x-y,t)g_n(y,t)\,\mathrm{d}y + X^{\sigma}S_1\bar{b}\bar{N}_{\lceil\beta\rceil}.$$

Let $f_1 * f_2$ is the convolution given by

$$f_1 * f_2(x) = \int_0^x f_1(x-y)f_2(y)\,\mathrm{d}y.$$

Therefore, the above inequality becomes

$$\frac{\partial g_n(x,t)}{\partial t} \le \frac{k}{2}(1+X)^{2\lambda}X^{\sigma}g_n * g_n(x,t) + S_1\bar{b}\bar{N}_{\lceil\beta\rceil}.$$

Integrating both sides with respect to t,

$$g_n(x,t) \le g(x,0) + \int_0^t \left[\frac{k}{2}(1+X)^{2\lambda}X^{\sigma}g_n * g_n(x,s) + S_1\bar{b}\bar{N}_{\lceil\beta\rceil}\right]\mathrm{d}s.$$

Let us consider $h(x,t)$ to be the function

$$h(x,t) = h_0 + \int_0^t \left[\frac{k}{2}(1+X)^{2\lambda}X^{\sigma}h * h(x,s) + h(x,s)\right]\mathrm{d}s, \quad 0 \le t \le T,\ 0 < x < \infty,$$

where $h_0 = \max_{\Pi(X_1,X_2,T)}\left\{g(x,0), S_1\bar{b}\bar{N}_{\lceil\beta\rceil}\right\}$.

First, taking the Laplace transform and then the inverse Laplace transform, we get the solution of the above integral equation as

$$h(x,t) = h_0\exp\left\{\frac{1}{2}h_0 xk(1+X)^{2\lambda}X^{\sigma}\left(e^t-1\right)+t\right\},$$

with $0 \le t \le T$ and $0 < x < \infty$.

Now, we claim that $h(x,t) \le g_n(x,t)$ for $(x,t) \in \Pi(X_1,X_2,T)$ and $\forall\ n \ge 1$.

So, for this, let us set the auxiliary function

$$h_\epsilon(x,t) = h_0 + \epsilon + \int_0^t \left[\frac{k}{2}(1+X)^{2\lambda}X^{\sigma}h_\epsilon * h_\epsilon(x,s) + h_\epsilon(x,s)\right]\mathrm{d}s, \quad \text{with} \quad (x,t)\in\Pi,\ \epsilon > 0.$$

Here,

$$h_\epsilon(x,0) = h_0 + \epsilon > h_0 \ge g(x,0), \qquad \text{for } X_1 \le x \le X_2. \tag{13}$$

We now prove our claim by the method of contradiction.

Let us assume that there exists a set D of points $(x,t) \in \Pi(X_1,X_2,T)$ on which $g_n(x,t) = h_\epsilon(x,t)$. From the relation (13), it is clear that D does not contain any points of the coordinate axes. We choose a point $(x_0,t_0) \in D$ such that the rectangle $Q = [X_1,x_0[\,\times\,[0,t_0[$ does not contain any points of D. Since h_ϵ, g_n are continuous, we have $g_n(x,t)$ to be strictly less than $h_\epsilon(x,t)$ in Q. So,

$$\begin{aligned}
g_n(x_0,t_0) &= h_\epsilon(x_0,t_0) \\
&> h_0 + \epsilon + \int_0^{t_0}\left[\frac{k}{2}(1+X)^{2\lambda}X^{\sigma}g_n * g_n(x_0,s) + S_1\bar{b}\bar{N}_{\lceil\beta\rceil}\right]\mathrm{d}s \\
&\quad \text{since, in } Q,\ g_n(x_0,t) < h_\epsilon(x_0,t), \quad \forall\ t\in[0,t_0] \\
&> g(x,0) + \epsilon + \int_0^{t_0}\left[\frac{k}{2}(1+X)^{2\lambda}X^{\sigma}g_n * g_n(x_0,s) + S_1\bar{b}\bar{N}_{\lceil\beta\rceil}\right]\mathrm{d}s \\
&\ge g_n(x_0,t_0) + \epsilon > g_n(x_0,t_0). \quad \text{A contradiction.}
\end{aligned}$$

This proves that D is empty and

$$\begin{aligned} g_n(x,t) &\leq h_\epsilon(x,t)\ \forall\ (x,t)\in\Pi(X_1,X_2,T) \text{ and } \forall\ n\geq 1\\ &= h_0+\epsilon+\int_0^t \left[\frac{k}{2}(1+X)^{2\lambda}X^\sigma h_\epsilon * h_\epsilon(x,s)+h_\epsilon(x,s)\right]\mathrm{d}s\\ &= (h_0+\epsilon)\exp\left\{\frac{k}{2}(1+X)^{2\lambda}X^\sigma(h_0+\epsilon)x\left(e^t-1\right)+t\right\}. \end{aligned}$$

Taking $\epsilon\to 0$ implies $g_n(x,t)<h(x,t),\ \forall\ (x,t)\in\Pi(X_1,X_2,T),\ n\geq 1$. Thus,

$$g_n(x,t)\leq h_0\exp\left\{\frac{k}{2}h_0X^{1+\sigma}(1+X)^{2\lambda}\left(e^T-1\right)+T\right\}=L_1 \text{ (say)}$$

Therefore,

$$f_n(x,t)\leq L_1X^\sigma=L,\qquad \text{a constant.} \tag{14}$$

Hence, the sequence $\{f_n(x,t)\}$ is uniformly bounded on $\Pi(X_1,X_2,T)$.

Step II. We now establish the equicontinuity of $\{f_n(x,t)\}$ with respect to x in the rectangle $\Pi(X_1,X_2,T)$. Let us assume $X_1\leq x\leq x'\leq X_2$, and then, for each $n\geq 1$, we have

$$\begin{aligned} |f_n(x',t)-f_n(x,t)| \leq |f_0(x')-f_0(x)| + \frac{1}{2}\int_0^t \Bigg[\int_x^{x'} & K_n(x'-y,y)f_n(x'-y,s)f_n(y,s)\,\mathrm{d}y\\ &+\frac{1}{2}\int_0^x |K_n(x'-y,y)-K_n(x-y,y)|\,f_n(x'-y,s)f_n(y,s)\,\mathrm{d}y\\ &+\frac{1}{2}\int_0^x K_n(x-y,y)\,|f_n(x'-y,s)-f_n(x-y,s)|\,f_n(y,s)\,\mathrm{d}y\\ &+|f_n(x',s)-f_n(x,s)|\int_0^\infty K_n(x',y)f_n(y,s)\,\mathrm{d}y \qquad (15)\\ &+f_n(x,s)\int_0^\infty |K_n(x',y)-K_n(x,y)|\,f_n(y,s)\,\mathrm{d}y \qquad (16)\\ &+\int_{x'}^\infty |b(x',y)-b(x,y)|\,S_n(y)f_n(y,s)\,\mathrm{d}y \qquad (17)\\ &+\int_x^{x'} b(x,y)S_n(y)f_n(y,s)\,\mathrm{d}y\Bigg]\,\mathrm{d}s\\ &+\int_0^t [|S_n(x')-S_n(x)|\,f_n(x',s) \qquad (18)\\ &+S_n(x)\,|f_n(x',s)-f_n(x,s)|]\,\mathrm{d}s. \qquad (19) \end{aligned}$$

According to the construction, the sequence of continuous kernels $\{K_n,S_n\}_{n=1}^\infty$ is equicontinuous over the rectangles $[X_1,X_2]\times[z_1,z_2]$, $z_1,z_2>0$, and $[X_1,X_2]$, respectively, and we have $b(x,y)$ continuous over the rectangle $[X_1,X_2]\times[z_1,z_2]$.
We aim to show that when $|x'-x|$ is small enough, then the left-hand side of Eq. (19) is small too. Corresponding to arbitrary $\epsilon>0$, there exists an $\delta(\epsilon)$ and $0<\delta(\epsilon)<\epsilon$ with

$$\sup_{|x'-x|<\delta}|f_0(x')-f_0(x)|<\epsilon,$$

$$\sup_{|x'-x|<\delta}|K_n(x',y)-K_n(x,y)|<\epsilon,$$

$$\sup_{|x'-x|<\delta} |S_n(x') - S_n(x)| < \epsilon,$$

and

$$\sup_{|x'-x|<\delta} |b(x',y) - b(x,y)| < \epsilon.$$

The above inequalities hold uniformly with respect to $n \geq 1$ and $z_1 \leq y \leq z_2$. The way of choosing these z_1 and z_2 is described below. We introduce the modulus of continuity as

$$\omega_n(t) = \sup_{|x'-x|<\delta} |f_n(x',t) - f_n(x,t)|, \quad X_1 \leq x, x' \leq X_2.$$

For the Eq. (16), we first do the breakup $\int_0^\infty = \int_0^{z_1} + \int_{z_1}^{z_2} + \int_{z_2}^\infty$ and proceed as follows:

$$\begin{aligned}
\int_0^\infty |K_n(x',y) - K_n(x,y)| f_n(y,s)\,\mathrm{d}y &\leq \int_0^{z_1} |K_n(x',y) - K_n(x,y)| f_n(y,s)\,\mathrm{d}y \\
&\quad + \int_{z_1}^{z_2} |K_n(x',y) - K_n(x,y)| f_n(y,s)\,\mathrm{d}y \\
&\quad + \int_{z_2}^{\infty} |K_n(x',y) - K_n(x,y)| f_n(y,s)\,\mathrm{d}y \\
&\leq \epsilon \bar{N}_0 + 2k \int_0^{z_1} \frac{(1+x)^\lambda (1+y)^\lambda}{(xy)^\sigma} f_n(y,s)\,\mathrm{d}y \\
&\quad + k \int_{z_2}^{\infty} \left| \frac{(1+x')^\lambda (1+y)^\lambda}{(x')^\sigma y^\sigma} - \frac{(1+x)^\lambda (1+y)^\lambda}{x^\sigma y^\sigma} \right| f_n(y,s)\,\mathrm{d}y \\
&\leq \epsilon \bar{N}_0 + 2k \int_0^{z_1} \frac{1 + x^\lambda + y^\lambda + (xy)^\lambda}{(xy)^\sigma} f_n(y,s)\,\mathrm{d}y \\
&\quad + k \int_{z_2}^{\infty} \frac{(1+y)^\lambda}{y^\sigma} \left| \frac{(1+x')^\lambda}{(x')^\sigma} - \frac{(1+x)^\lambda}{x^\sigma} \right| f_n(y,s)\,\mathrm{d}y \\
&\leq \epsilon \bar{N}_0 + 2k \frac{1+x^\lambda}{x^\sigma} \int_0^{z_1} \frac{y^{\gamma-\sigma}}{y^\gamma} f_n(y,s)\,\mathrm{d}y \\
&\quad + 2k \frac{1}{x^\sigma} \int_0^{z_1} \frac{y^{k_1+(\lambda-\sigma)}}{y^{k_1}} f_n(y,s)\,\mathrm{d}y \\
&\quad + 2k x^{\lambda-\sigma} \int_0^{z_1} \frac{y^{k_1+(\lambda-\sigma)}}{y^{k_1}} f_n(y,s)\,\mathrm{d}y \\
&\quad + k \int_{z_2}^{\infty} \frac{(1+y^\lambda)}{y^\sigma} \left| \left(1 + \frac{1}{x'}\right)^\sigma (1+x')^{\lambda-\sigma} \right. \\
&\quad \left. - \left(1 + \frac{1}{x}\right)^\sigma (1+x)^{\lambda-\sigma} \right| f_n(y,s)\,\mathrm{d}y.
\end{aligned}$$

We choose $k_1 = \gamma$ only when $\lambda - \sigma = 0$ and $k_1 = 0$ when $0 < \lambda - \sigma \leq 1$.

$$\begin{aligned}\int_0^\infty |K_n(x',y) - K_n(x,y)|\, f_n(y,s)\,\mathrm{d}y &\leq \epsilon \bar{N}_0 + 2k\frac{(1+X)z_1^{\gamma-\sigma}}{X^\sigma}\int_0^{z_1}\frac{1}{y^\gamma}f_n(y,s)\,\mathrm{d}y \\ &\quad + 2kX^\sigma z_1^{k_1+(\lambda-\sigma)}\int_0^{z_1}\frac{1}{y^{k_1}}f_n(y,s)\,\mathrm{d}y \\ &\quad + 2kX^{\lambda-\sigma} z_1^{k_1+(\lambda-\sigma)}\int_0^{z_1}\frac{1}{z^{k_1}}f_n(y,s)\,\mathrm{d}y \\ &\quad + k\delta\left(1+\frac{1}{x}\right)\int_{z_2}^\infty \left(y^{\lambda-\sigma}+\frac{1}{y^\sigma}\right)f_n(y,s)\,\mathrm{d}y \\ &\leq \epsilon\bar{N}_0 + 2k(1+X)X\left[\bar{N}_\gamma z_1^{\gamma-\sigma} + 2\bar{N}_{k_1} z_1^{k_1+(\lambda-\sigma)}\right] \\ &\quad + k\delta\,(1+X)\int_{z_2}^\infty\left(y^{\lambda-\sigma}+\frac{1}{y^\sigma}\right)f_n(y,s)\,\mathrm{d}y. \end{aligned} \tag{20}$$

Let us take $\phi(x)$ be a nonnegative and measurable function and $\psi(x)$ is positive and nondecreasing for $x > 0$, then we have the relation

$$\int_z^\infty \phi(x)\,\mathrm{d}x \leq \frac{1}{\psi(z)}\int_0^\infty \phi(x)\psi(x)\,\mathrm{d}x,\ \ z > 0, \tag{21}$$

if the integrals exist and are finite.

Now, in the relation (21), we put $\phi(x) = f_n(x),\ \psi(x) = x$ and $\phi(x) = x^{\lambda-\sigma}f_n(x),\ \psi(x) = x$, respectively, to get

$$\int_{z_2}^\infty \frac{1}{y^\sigma}f_n(y,s)\,\mathrm{d}y \leq \frac{1}{z_2^{1+\sigma}}\bar{N}_0,$$

and

$$\int_{z_2}^\infty y^{\lambda-\sigma}f_n(y,s)\,\mathrm{d}y \leq \frac{1}{z_2}\bar{N}_1.$$

Using these relations in Eq. (20), we get

$$\begin{aligned}\int_0^\infty |K_n(x',y) - K_n(x,y)|\, f_n(y,s)\,\mathrm{d}y \leq\ & \epsilon\bar{N}_0 + 2k(1+X)X^\sigma\left[z_1^{\gamma-\sigma}\bar{N}_\gamma + 2z_1^{k_1+(\lambda-\sigma)}\bar{N}_{k_1}\right] \\ & + k\delta\,(1+X)\frac{1}{z_2}\left[\frac{\bar{N}_0}{z_2^\sigma}+\bar{N}_1\right]. \end{aligned} \tag{22}$$

Now, we choose this z_1 and z_2 such that $z_1^{\gamma-\sigma}\bar{N}_\gamma \leq \epsilon$, $z_1^{k_1+(\lambda-\sigma)}\bar{N}_{k_1} \leq \epsilon$, $\frac{1}{z_2^{1+\sigma}}\bar{N}_0 \leq \epsilon$ and $\frac{1}{z_2}\bar{N}_1 \leq \epsilon$.

Hence, Eq. (22) implies

$$\int_0^\infty |K_n(x',y) - K_n(x,y)|\, f_n(y,s)\,\mathrm{d}y \leq \epsilon\left[\bar{N}_0 + 6k(1+X)X + 2k(1+X)\right]. \tag{23}$$

For the Eq. (15), we do the breakup $\int_0^\infty = \int_0^{z_1} + \int_{z_1}^{z_2} + \int_{z_2}^\infty$ of the integral over y, like as we have done in the case of (16). The second integral is a finite term and the first, and the third integrals are small quantity due to Eq. (23). Hence,

$$\begin{aligned}|f_n(x',s) - f_n(x,s)|\int_0^\infty K_n(x',y)f_n(y,s)\,\mathrm{d}y &\leq k\omega_n(s)\int_0^\infty \frac{(1+x)^\lambda(1+y)^\lambda}{(xy)^\sigma}f(y,s)\,\mathrm{d}y \\ &\leq M_1\omega_n(s). \end{aligned} \tag{24}$$

Now, Eq. (17) implies

$$\int_{x'}^{\infty} |b(x',y)-b(x,y)|\, S_n(y)f_n(y,s)\,\mathrm{d}y \leq \epsilon\bar{N}_{\lceil\beta\rceil}+2S_1\int_{z_2}^{\infty} y^{\beta}b(x,y)f_n(y,s)\,\mathrm{d}y$$
$$\leq \epsilon\bar{N}_{\lceil\beta\rceil}+2S_1\bar{b}\int_{z_2}^{\infty} y^{\beta}f_n(y,s)\,\mathrm{d}y$$
$$\leq \epsilon(\bar{N}_{\lceil\beta\rceil}+2\bar{b}S_1),\ \text{[by previous arguments]}. \tag{25}$$

The integrand in

$$\int_0^t\int_x^{x'} b(x,y)S_n(y)f_n(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

is continuous with compact support in $\Pi(X_1,X_2,T)$, $\forall\ n\geq 1$. So, when

$$|x'-x|<\delta \Rightarrow \int_0^t\int_x^{x'} b(x,y)S_n(y)f_n(y,s)\,\mathrm{d}y\,\mathrm{d}s \text{ is a small quantity.}$$

We already have

$$\sup_{|x'-x|<\delta}|S_n(x')-S_n(x)|<\epsilon$$

and using (14), Eq. (18) gives

$$\int_0^t |S_n(x')-S_n(x)|\, f_n(x',s)\,\mathrm{d}s<\epsilon LT. \tag{26}$$

The last equation in (19) implies

$$\int_0^t S_n(x)\,|f_n(x',s)-f_n(x,s)|\,\mathrm{d}s\leq S_1X^{\beta}\int_0^t\omega_n(s)\,\mathrm{d}s. \tag{27}$$

For the first integral in relation to (19), we have

$$\frac{1}{2}\int_x^{x'} K_n(x'-y,y)f_n(x'-y,s)f_n(y,s)\,\mathrm{d}y\leq \frac{1}{2}L^2k(1+X)^{2\lambda}\int_x^{x'}\frac{1}{(x'-y)^{\sigma}y^{\sigma}}\,\mathrm{d}y$$
$$\leq M_2\,|x'-x|\,,\ M_2 \text{ is a constant.}$$

Hence, it is a small quantity. Similarly, for the second and the third terms in (19) whose integrals are over the finite range, we can show that they are small quantities.

Using the relations (23), (24), (25), (26), and (27) in the Eq. (19), we get

$$\omega_n(t)\leq \left[L\left(\bar{N}_0+6k(1+X)X+2k(1+X)\right)+(\bar{N}_{\lceil\beta\rceil}+2\bar{b}S_1)+LT\right]\epsilon$$
$$+\ (M_1+S_1X^{\beta})\int_0^t\omega_n(s)\,\mathrm{d}s$$
$$\leq M_3\cdot\epsilon+M_4\int_0^t\omega_n(s)\,\mathrm{d}s,$$

where $M_3=\left[L\left(\bar{N}_0+6k(1+X)X+2k(1+X)\right)+(\bar{N}_{\lceil\beta\rceil}+2\bar{b}S_1)+LT\right]$ and $M_4=[M_1+S_1X^{\beta}]$ and these M_3 and M_4 are constants independent of n and ϵ.

By applying Gronwall's inequality, we get

$$\omega_n(t)\leq M_3\cdot\epsilon\exp(M_4T)\leq M_5\cdot\epsilon. \tag{28}$$

Hence, the equicontinuity with respect to x on $\Pi(X_1,X_2,T)$ is obtained.

Step III. We show the equicontinuity of $\{f_n(x,t)\}$ with respect to t over the rectangle $\Pi(X_1, X_2, T)$. Let us choose $0 \le t \le t' \le T,\ n \ge 1$. By definition of equicontinuity, corresponding to an arbitrary $\epsilon > 0$, there exists an $\delta(\epsilon) > 0$ for which

$$|f_n(x,t') - f_n(x,t)| < \epsilon, \quad \text{whenever } |t' - t| < \delta.$$

From Eq. (5), we get the following inequality

$$\begin{aligned} |f_n(x,t') - f_n(x,t)| \le \int_t^{t'} \Bigg[& \frac{1}{2}\int_0^x K_n(x-y,y) f_n(x-y,s) f_n(y,s)\,\mathrm{d}y \\ & + f_n(x,s)\int_0^\infty K_n(x,y) f_n(y,s)\,\mathrm{d}y \\ & + \int_x^\infty b(x,y) S_n(y) f_n(y,s)\,\mathrm{d}y + f_n(x,s) S_n(x)\Bigg]\,\mathrm{d}s. \end{aligned} \tag{29}$$

So if $|t' - t| < \delta(\epsilon)$, then we need to show that the left-hand side of (29) is small too.

For the first integral

$$\begin{aligned} \frac{1}{2}\int_0^x K_n(x-y,y) f_n(x-y,s) f_n(y,s)\,\mathrm{d}y &\le \frac{k}{2}(1+X)^{2\lambda}\int_0^x \frac{1}{(x-y)^\sigma y^\sigma} f_n(x-y,t) f_n(y,t)\,\mathrm{d}y \\ &\le \frac{k}{2}(1+X)^{2\lambda} L^2 \int_0^x \frac{1}{(x-y)^\sigma y^\sigma}\,\mathrm{d}y. \end{aligned}$$

[since, $\{f_n\}s$ are uniformly bounded by L over $\Pi(X_1, X_2, T)$].

By some computation, we get the integral

$$\int_0^x \frac{1}{(x-y)^\sigma y^\sigma}\,\mathrm{d}y$$

to be well defined and convergent (say M_6) in $\Pi(X_1, X_2, T)$. Therefore,

$$\frac{1}{2}\int_0^x K_n(x-y,y) f_n(x-y,s) f_n(y,s)\,\mathrm{d}y \le \frac{k}{2}(1+X)^{2\lambda} L^2 M_6 = \text{constant}.$$

For the second integral using the relation (24), we get

$$f_n(x,s)\int_0^\infty K_n(x,y) f_n(y,s)\,\mathrm{d}y \le L M_1 = \text{ constant}.$$

For the third integral

$$\begin{aligned} \int_t^{t'}\int_x^\infty S_n(y) b(x,y) f_n(y,s)\,\mathrm{d}y &\le \int_0^{t'}\int_x^\infty S_n(y) b(x,y) f_n(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \int_0^{t}\int_x^\infty S_n(y) b(x,y) f_n(y,s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

In $\Pi(X_1, X_2, T)$, we have

$$\begin{aligned} \int_x^\infty S_n(x) b(x,y) f_n(y,t)\,\mathrm{d}y &\le \bar{b} S_1 \int_0^\infty y^\beta f_n(y,s)\,\mathrm{d}y \\ &\le \bar{b} S_1 \bar{N}_{\lceil \beta \rceil}. \end{aligned}$$

So, we get when $|t' - t| < \delta$,

$$\int_t^{t'}\int_x^\infty S_n(y) b(x,y) f_n(y,s)\,\mathrm{d}y\,\mathrm{d}s \le \epsilon \bar{b} S_1 \bar{N}_{\lceil \beta \rceil}\delta.$$

The fourth term

$$S_n(x) f_n(x,s) \leq L S_1 X^{\beta}.$$

Combining all these and putting in (29), we get

$$|f_n(x,t') - f_n(x,t)| \leq \int_t^{t'} \left[\frac{k}{2} (1+X)^{2\lambda} L^2 M_6 + L M_1 + \bar{b} S_1 \bar{N}_{\lceil \beta \rceil} + L S_1 X^{\beta} \right] \mathrm{d}s \leq M_7 |t'-t|, \tag{30}$$

where $0 \leq t \leq t' \leq T,\ n \geq 1$ and $M_7 = \left[\frac{k}{2} (1+X)^{2\lambda} L^2 M_6 + L M_1 + \bar{b} S_1 \bar{N}_{\lceil \beta \rceil} + L S_1 X^{\beta} \right]$. The constant M_7 is independent of $n,\ \epsilon$ and hence $\{f_n\}_{n=1}^{\infty}$ is equicontinuous with respect to the variable t on $\Pi(X_1, X_2, T)$.

Thus, from the Eqs. (28) and (30) together, we can conclude that

$$\sup_{|x'-x|<\delta, |t'-t|<\delta} |f_n(x',t') - f_n(x,t)| \leq (M_5 + M_7)\epsilon \text{ for } X_1 \leq x, x' \leq X_2,\ 0 \leq t, t' \leq T, \tag{31}$$

where M_5 and M_7 are constants independent of n and ϵ.

Therefore, the relations (14), (31) along with the Arzelà–Ascoli theorem [1,8], we get that there exists a sequence $\{f_n\}_{n=1}^{\infty}$, which is relatively compact in the uniform convergence topology of continuous functions on each rectangle $\Pi(X_1, X_2, T)$.

Proof of Theorem 2.1

By means of diagonal method, we select a subsequence $\{f_p\}_{p=1}^{\infty}$ from $\{f_n\}_{n=1}^{\infty}$ converging uniformly on each compact set in Π to a continuous, nonnegative function f and satisfies Eqs. (11) and (12). Let us consider the integrals $\int_{z_1}^{z_2} \left(x^{j_1} + \frac{1}{x^{j_2}}\right) f(x,t)\, \mathrm{d}x$ for $0 \leq j_1 \leq r_1,\ 0 < j_2 < 1$.

So, there exists $p \geq 1$ such that for all $\epsilon > 0$,

$$\int_{z_1}^{z_2} \left(x^{j_1} + \frac{1}{x^{j_2}}\right) f(x,t)\, \mathrm{d}x \leq \int_{z_1}^{z_2} \left(x^{j_1} + \frac{1}{x^{j_2}}\right) f_p(x,t)\, \mathrm{d}x + \epsilon, \quad \text{for } 0 \leq j_1 \leq r_1,\ 0 < j_2 < 1. \tag{32}$$

In (32), all of z_1, z_2, and ϵ are arbitrary, so

$$\int_0^{\infty} \left(x^{j_1} + \frac{1}{x^{j_2}}\right) f(x,t)\, \mathrm{d}x \leq \bar{N}_{j_1} + \bar{N}_{j_2},\ 0 \leq j_1 \leq r_1,\ \ 0 < j_2 < 1. \tag{33}$$

We now show that the function $f(x,t)$ is a solution to the initial-value problem (1), (2). For this, we make some rearrangement of the terms in Eq. (5). We replace $K_n,\ S_n$ and f_n in (5) by $K_p - K + K,\ S_p - S + S$

and $f_p - f + f$, respectively, and obtain

$$
\begin{aligned}
(f_p - f)(x,t) + f(x,t) =& f_0(x) + \int_0^t \Big[\int_0^x \frac{1}{2}(K_p - K)(x-y,y) f_p(x-y,s) f_p(y,s)\,\mathrm{d}y \\
&+ \int_0^x \frac{1}{2} K(x-y,y)(f_p(x-y,s) - f(x-y,s)) f_p(y,s)\,\mathrm{d}y \\
&+ \int_0^x \frac{1}{2} K(x-y,y)(f_p(y,s) - f(y,s)) f(x-y,s)\,\mathrm{d}y \\
&+ \int_0^x \frac{1}{2} K(x-y,y) f(x-y,s) f(y,s)\,\mathrm{d}y \\
&- f_p(x,s) \int_0^\infty (K_p - K)(x,y) f_p(y,s)\,\mathrm{d}y \\
&- (f_p - f)(x,s) \int_0^\infty K(x,y) f_p(y,s)\,\mathrm{d}y \\
&- f(x,s) \int_0^\infty K(x,y)(f_p - f)(y,s)\,\mathrm{d}y \\
&+ f(x,s) \int_0^\infty K(x,y) f(y,s)\,\mathrm{d}y \\
&+ \int_x^\infty b(x,y)(S_p - S)(y) f_p(y,s)\,\mathrm{d}y \\
&+ \int_x^\infty b(x,y) S(y)(f_p - f)(y,s)\,\mathrm{d}y \\
&+ \int_x^\infty b(x,y) S(y) f(y,s)\,\mathrm{d}y - (S_p - S)(x) f_p(x,t) \Big]\,\mathrm{d}s \\
&- \int_0^t [S(x)(f_p - f)(x,s) - S(x) f(x,s)]\,\mathrm{d}s. \qquad (34)
\end{aligned}
$$

Passing to the limit $p \to \infty$ in (34), we can get that the terms involving integrals over the infinite range tend to zero due to the estimates for their tails. To establish this argument, we proceed as follows

$$
\begin{aligned}
\left| \int_0^\infty (K_p - K)(x,y) f_p(y,s)\,\mathrm{d}y \right| \le & \left| \int_0^{z_1} (K_p - K)(x,y) f_p(y,s)\,\mathrm{d}y \right| \\
&+ \left| \int_{z_1}^{z_2} (K_p - K)(x,y) f_p(y,s)\,\mathrm{d}y \right| \\
&+ \left| \int_{z_2}^\infty (K_p - K)(x,y) f_p(y,s)\,\mathrm{d}y \right|
\end{aligned}
$$

and

$$
\left| \int_0^\infty (K_p - K)(x,y) f_p(y,s)\,\mathrm{d}y \right| \le M_8 \epsilon, \ [\text{ Using } (23)]. \qquad (35)
$$

Similarly,

$$
\left| \int_0^\infty K(x,y)(f_p - f)(y,s)\,\mathrm{d}y \right| \le M_8 \epsilon, \qquad (36)
$$

and using (25), we have

$$
\int_x^\infty b(x,y)(S_p - S)(y) f_p(y,s)\,\mathrm{d}y \le M_9 \epsilon,
$$

and

$$\int_x^\infty b(x,y)S(y)(f_p - f)(y,s)\,\mathrm{d}y \le M_9\epsilon.$$

The other difference terms involving their integrals over the finite integrals are convergent as the integral $\int_0^x K_n(x-y,y)f_n(x-y,s)f_n(y,s)\,\mathrm{d}y$ is convergent and hence finite (we have proved this earlier), and from the definition of $\epsilon-\delta$, we can easily show that those integrals tend toward zero as $p\to\infty$. Finally, we find that the function $f(x,t)$ is a solution of the problem (1), (2), which is obtained from Eq. (34) in the following form,

$$\begin{aligned} f(x,t) =& f_0(x) + \int_0^t \left[\frac{1}{2}\int_0^x K(x-y,y)f(x-y,s)f(y,s)\,\mathrm{d}y - f(x,s)\int_0^\infty K(x,y)f(y,s)\,\mathrm{d}y \right. \\ &\left. + \int_x^\infty b(x,y)S(x)f(y,s)\,\mathrm{d}y - S(x)f(x,s).\right]\mathrm{d}s. \end{aligned} \tag{37}$$

All the different terms involved in (34) tend toward zero and $f(x,t)$ is continuous. Using these, we can say that the right-hand side of (1) is a continuous function in Π. So, on the differentiation of (37) with respect to t, it establishes that $f(x,t)$ is a continuous and differentiable solution of (1), (2) in the space $\Omega^+_{r_1,r_2}(T)$ [by relation (33)]. This proves the existence of the solution to (1), (2) and hence the Theorem 2.1.

Remark 2.3. Here, we have obtained a strong solution to the problem (1), (2). Hence, the solution is differentiable with respect to t in $\Omega^+_{r_1,r_2}(T)$, whereas in [3], existence of weak solutions has been obtained for a smaller class of coagulation and fragmentation kernels.

3. Mass conservation

Theorem 3.1. *Let the conditions of Theorem 2.1 hold good, then the solution of Eqs. (1), (2) satisfies the mass conservation law.*

Proof. We are now ready to prove the mass conservation law similar to Eq. (7),

$$N_1 = \int_0^\infty x f(x,t)\,\mathrm{d}x = \bar{N}_1 = \text{ constant}, \ \ \forall\, t\ge 0.$$

Integrating (1) with a weight x, we obtain

$$\begin{aligned} \frac{\mathrm{d}N_1}{\mathrm{d}t} =& \int_0^\infty x\left[\frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,\mathrm{d}y - \int_0^\infty K(x,y)f(x,t)f(y,t)\,\mathrm{d}y \right. \\ &\left. + \int_x^\infty S(y)b(x,y)f(y,t)\,\mathrm{d}y - S(x)f(x,t)\right]\mathrm{d}x. \end{aligned} \tag{38}$$

In Eq. (38), for the last two terms involving the fragmentation kernels, we have

$$\int_0^\infty x\left[\int_x^\infty S(y)b(x,y)f(y,t)\,\mathrm{d}y - S(x)f(x,t)\right]\mathrm{d}x,$$

changing the order of integration in the first integral, and using (3), we get

$$\begin{aligned} &\int_0^\infty yS(y)f(y,t)\,\mathrm{d}y - \int_0^\infty yS(y)f(y,t)\,\mathrm{d}y \\ &= 0 \text{ [the moment } N_{\lceil\beta+1\rceil} \text{ is bounded].} \end{aligned}$$

Therefore, (38) turns out to be in the form below

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \int_0^\infty x\left[\frac{1}{2}\int_0^x K(x-y,y)f(x-y,t)f(y,t)\,\mathrm{d}y - \int_0^\infty K(x,y)f(x,t)f(y,t)\,\mathrm{d}y\right].$$

Changing the order of integration of the first integral and then replacing $x-y=x',\ y=y'$ and rechanging the order of integration, we get

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{1}{2}\int_0^\infty\int_0^\infty (x+y)K(x,y)f(x,t)f(y,t)\,\mathrm{d}y\,\mathrm{d}x - \int_0^\infty\int_0^\infty xK(x,y)f(x,t)f(y,t)\,\mathrm{d}y\,\mathrm{d}x.$$

The integral

$$\int_0^\infty\int_0^\infty xK(x,y)f(x,t)f(y,t)\,\mathrm{d}y\,\mathrm{d}x$$

is bounded due to the bound of $K(x,y)$ and the boundedness of the second moment N_2. So, using the symmetric nature of $K(x,y)$, we can conclude the following

$$\frac{1}{2}\int_0^\infty\int_0^\infty (x+y)K(x,y)f(x,t)f(y,t)\,\mathrm{d}y\,\mathrm{d}x - \int_0^\infty\int_0^\infty xK(x,y)f(x,t)f(y,t)\,\mathrm{d}y\,\mathrm{d}x = 0$$

, which implies $\frac{\mathrm{d}N_1}{\mathrm{d}t}=0$. Hence, $N_1=\int_0^\infty xf(x,t)\,\mathrm{d}x$ is bounded (say $\bar{N}_1$) proving the mass conservation law.

4. Uniqueness theorem

We have already proved the existence of the solution for the problems (1), (2) in Sect. 2. The next question left is whether the solution is unique or not? In this section, we study the uniqueness of the solution to (1), (2). In Sect. 2, we have defined our considered class of domains as $\Omega^+_{r_1,r_2}(T)$ for $r_1 \geq 1,\ 0 < r_2 < 1$. Now, in this section, we aim to prove our uniqueness theorem for those class of selection functions that satisfy certain restriction.

Firstly, let us redefine the spaces,

$$\Omega_{r_1,r_2}(T) = \left\{\text{all continuous functions } f(x,t) : \sup_{0\leq t\leq T}\int_0^\infty\left(x^{r_1}+\frac{1}{x^{r_2}}\right)|f(x,t)|\,\mathrm{d}x = \|f\|_{r_1,r_2} < \infty\right\},$$

where $r_1 \geq 1,\ \frac{1}{2} < r_2 < 1$.

Like before, here also we denote the cone of the nonnegative functions in $\Omega_{r_1,r_2}(T)$ as $\Omega^+_{r_1,r_2}(T)$. Now, to obtain the uniqueness of the solutions to (1), (2), we are going to prove the following theorem:

Theorem 4.1. *Let the functions $K(x,y)$, $b(x,y)$ be continuous, nonnegative over $]0,\infty[\times]0,\infty[$, $S(x)$ be continuous and nonnegative over $]0,\infty[$, and $K(x,y)$ is symmetric $\forall\ x,y\in]0,\infty[$. Suppose*

- (i) $\forall\ x,y \in]0,\infty[,\ K(x,y) \leq k\frac{(1+x+y)^\lambda}{(xy)^\sigma}$ *with $k>0$ a constant, $\sigma\in\left[0,\frac{1}{2}\right]$ and $\lambda-\sigma\in[0,1]$,*
- (ii) $S(x)\leq S_1x^\beta$, $\forall\ x\in]0,\infty[$ *where $S_1(>0)$ a constant, $0<\beta\leq 1$,*
- (iii) $S_0x^\alpha < S(x)$, *where $S_0(>0)$ a constant, $0<\alpha\leq 1$, $x\geq x_0$ where $x_0(\geq 1)$ is a large number,*
- (iv) *for some real number γ such that $0<\gamma<1$ satisfying $\int_0^y \frac{1}{x^\gamma}b(x,y)\,dx \leq \frac{N_0}{y^\gamma}$ where $N_0(>0)$ is a constant,*
- (v) $\lim_{y\to\infty}\sup_{x\in[x_1,x_2]} b(x,y) \leq \bar{b}$, $\forall\ 0<x_1<x_2<\infty$ *and $\bar{b}$ is a constant,*

and let the initial data satisfy $f_0(x)\in\Omega^+_{r_1,r_2}(0)$, then the problem (1), (2) has a unique solution in $\Omega^+_{r_1,r_2}(T)$.

Proof. Let us consider u_1 and u_2 be two solutions to (1), (2) on $[0,T]$, where $T>0$, with $u_1(x,0)=u_2(x,0)$ and we set $U(x,t)=u_1(x,t)-u_2(x,t)$. We define

$$M(t)=\int_0^\infty \left(x+\frac{1}{x^{k_2}}\right)|U(x,t)|\;\mathrm{d}x$$

where we choose k_2 to be $0<k_2\le \min\{r_2-\sigma,\alpha\}$ and satisfying condition (iv) of Theorem 4.1. □

By our construction, $U(x,t)$ is absolutely continuous over $[0,T]$ and for $x\in]0,\infty[$, and therefore, we can say that $U(x,t)$ satisfies the Eq. (1). So, we find the derivative of the solutions as

$$\begin{aligned}\frac{\partial U(x,t)}{\partial t}&=\frac{1}{2}\int_0^x K(x-y,y)\left\{u_1(x-y,t)u_1(y,t)-u_2(x-y,t)u_2(y,t)\right\}\,\mathrm{d}y\\&\quad-\int_0^\infty K(x,y)\left\{u_1(x,t)u_1(y,t)-u_2(x,t)u_2(y,t)\right\}\,\mathrm{d}y\\&\quad+\int_x^\infty b(x,y)S(y)\left\{u_1(y,t)-u_2(y,t)\right\}\,\mathrm{d}y-S(x)\left\{u_1(x,t)-u_2(x,t)\right\}.\end{aligned}\tag{39}$$

Let for $t\in\mathbb{R}$, we define

$$\operatorname{sgn}(t)=\begin{cases}1, & \text{when}\quad t>0,\\0, & \text{when}\quad t=0,\\-1, & \text{when}\quad t<0.\end{cases}$$

and

$$\frac{\mathrm{d}|P(t)|}{\mathrm{d}t}=\operatorname{sgn}(P(t))\frac{\mathrm{d}}{\mathrm{d}t}P(t).$$

Multiplying both sides of (39) by $\left(x+\frac{1}{x^{k_2}}\right)$ and integrating with respect to x from 0 to ∞, we get

$$\begin{aligned}M(t)=\int_0^t\int_0^\infty&\left(x+\frac{1}{x^{k_2}}\right)\operatorname{sgn}\left(U(x,s)\right)\\&\times\left[\frac{1}{2}\int_0^x K(x-y,y)\left\{u_1(x-y,s)u_1(y,s)-u_2(x-y,s)u_2(y,s)\right\}\,\mathrm{d}y\right.\qquad(40)\\&-\int_0^\infty K(x,y)\left\{u_1(x,s)u_1(y,s)-u_2(x,s)u_2(y,s)\right\}\,\mathrm{d}y\qquad(41)\\&\left.+\int_x^\infty b(x,y)S(y)\left\{u_1(y,s)-u_2(y,s)\right\}\,\mathrm{d}y-S(x)\left\{u_1(x,s)-u_2(x,s)\right\}\right]\,\mathrm{d}x\,\mathrm{d}s.\qquad(42)\end{aligned}$$

For the first integral (40), we have

$$\frac{1}{2}\int_0^\infty\int_0^x\left(x+\frac{1}{x^{k_2}}\right)\operatorname{sgn}\left(U(x,s)\right)K(x-y,y)\left\{u_1(x-y,s)u_1(y,s)-u_2(x-y,s)u_2(y,s)\right\}\,\mathrm{d}y\,\mathrm{d}x.$$

Changing the order of integration and then substituting $x-y=x',\ y=y'$ and rechanging the order of integration,

$$\begin{aligned}\frac{1}{2}\int_0^\infty\int_0^x&\left((x+y)+\frac{1}{(x+y)^{k_2}}\right)\operatorname{sgn}\left(U(x+y,s)\right)K(x,y)\\&\times\left\{u_1(x,s)u_1(y,s)-u_2(x,s)u_2(y,s)\right\}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Putting this relation in (42), we get

$$M(t) = \int_0^t \int_0^\infty \int_0^\infty \left[\left(\frac{1}{2}(x+y) + \frac{1}{2(x+y)^{k_2}}\right) \operatorname{sgn}(U(x+y,s)) - \left(x + \frac{1}{x^{k_2}}\right) \operatorname{sgn}(U(x,s))\right] \times K(x,y)\{u_1(x,s)u_1(y,s) - u_2(x,s)u_2(y,s)\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \tag{43}$$

$$+ \int_0^t \int_0^\infty \left(x + \frac{1}{x^{k_2}}\right) \operatorname{sgn}(U(x,s)) \left[\int_x^\infty b(x,y)S(y)\{u_1(y,s) - u_2(y,s)\}\,\mathrm{d}y - S(x)\{u_1(x,s) - u_2(x,s)\}\right] \mathrm{d}x\,\mathrm{d}s. \tag{44}$$

Using the symmetry of $K(x,y)$, by interchanging the roles of x and y and by changing the order of integration, we get the following identity

$$\int_0^\infty \int_0^\infty \left(x + \frac{1}{x^{k_2}}\right) \operatorname{sgn}(U(x,s))\, K(x,y)\{u_1(x,s)u_1(y,s) - u_2(x,s)u_2(y,s)\}\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^\infty \int_0^\infty \left(y + \frac{1}{y^{k_2}}\right) \operatorname{sgn}(U(y,s))\, K(x,y)\{u_1(x,s)u_1(y,s) - u_2(x,s)u_2(y,s)\}\,\mathrm{d}y\,\mathrm{d}x. \tag{45}$$

Therefore, using (45), we can rewrite the following Eq. as

$$\int_0^t \int_0^\infty \int_0^\infty \left(\frac{1}{2}(x+y) + \frac{1}{2(x+y)^{k_2}} - x - \frac{1}{x^{k_2}}\right) \times K(x,y)\{u_1(x,s)u_1(y,s) - u_2(x,s)u_2(y,s)\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s$$
$$= \frac{1}{2}\int_0^t \int_0^\infty \int_0^\infty \left[(x+y) + \frac{1}{(x+y)^{k_2}} - \left(x + \frac{1}{x^{k_2}}\right) - \left(y + \frac{1}{y^{k_2}}\right)\right] \times K(x,y)\{u_1(x,s)u_1(y,s) - u_2(x,s)u_2(y,s)\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s.$$

So, we rewrite Eqs. (43), (44) as follows

$$M(t) = \frac{1}{2}\int_0^t \int_0^\infty \int_0^\infty \left[\left((x+y) + \frac{1}{(x+y)^{k_2}}\right) \operatorname{sgn}(U(x+y,s)) - \left\{\left(x + \frac{1}{x^{k_2}}\right) \operatorname{sgn}(U(x,s)) + \left(y + \frac{1}{y^{k_2}}\right) \operatorname{sgn}(U(y,s))\right\}\right] \times K(x,y)\{u_1(x,s)u_1(y,s) - u_2(x,s)u_2(y,s)\}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \tag{46}$$

$$+ \int_0^t \int_0^\infty \left(x + \frac{1}{x^{k_2}}\right) \operatorname{sgn}(U(x,s)) \left[\int_x^\infty b(x,y)S(y)\{u_1(y,s) - u_2(y,s)\}\,\mathrm{d}y - S(x)\{u_1(x,s) - u_2(x,s)\}\right] \mathrm{d}x\,\mathrm{d}s. \tag{47}$$

For $x, y \geq 0$ and $t \in [0,T]$, we define the function w by

$$w(x,y) = \left[\left((x+y) + \frac{1}{(x+y)^{k_2}}\right) \operatorname{sgn}(U(x+y,s)) - \left\{\left(x + \frac{1}{x^{k_2}}\right) \operatorname{sgn}(U(x,s)) + \left(y + \frac{1}{y^{k_2}}\right) \operatorname{sgn}(U(y,s))\right\}\right].$$

We have $u_1U + u_2U = u_1(u_1 - u_2) + u_2(u_1 - u_2)$ and use this in (46),

$$
\begin{aligned}
M(t) &= \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty w(x,y)K(x,y)[u_1(x,s)U(y,s) + u_2(x,s)U(y,s)]\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^t\int_0^\infty\left(x + \frac{1}{x^{k_2}}\right)\operatorname{sgn}(U(x,s))\left[\int_x^\infty b(x,y)S(y)U(y,s)\,\mathrm{d}y - S(x)U(x,t)\right]\mathrm{d}x\,\mathrm{d}s \\
&= \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty w(x,y)K(x,y)u_1(x,s)U(y,s)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \qquad (48)\\
&\quad + \frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty w(x,y)K(x,y)u_2(x,s)U(y,s)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \qquad (49)\\
&\quad + \int_0^t\int_0^\infty\left(x + \frac{1}{x^{k_2}}\right)\left[\int_x^\infty b(x,y)S(y)U(y,s)\,\mathrm{d}y - S(x)U(x,s)\right] \\
&\quad \times \operatorname{sgn}(U(x,s))\,\mathrm{d}x\,\mathrm{d}s. \qquad (50)
\end{aligned}
$$

We have the relation

$$
\frac{1}{(x+y)^p} \le \frac{1}{x^p} + \frac{1}{y^p}, \qquad \text{if } p > 0,
$$

and taking in account that for all $t_1, t_2 \in \mathbb{R}$ that $\operatorname{sgn}(t_1)\operatorname{sgn}(t_2) = \operatorname{sgn}(t_1t_2)$ and $|t_1| = t_1\operatorname{sgn}(t_1)$, we can estimate

$$
\begin{aligned}
w(x,y)U(y,t) &= \left[\left((x+y) + \frac{1}{(x+y)^{k_2}}\right)\operatorname{sgn}(U(x+y,s))\right. \\
&\qquad \left. - \left\{\left(x + \frac{1}{x^{k_2}}\right)\operatorname{sgn}(U(x,s)) + \left(y + \frac{1}{y^{k_2}}\right)\operatorname{sgn}(U(y,s))\right\}\right] \\
&\le \left[\left(x + y + \frac{1}{(x+y)^{k_2}}\right) - \left(x + \frac{1}{x^{k_2}}\right) - \left(y + \frac{1}{y^{k_2}}\right)\right]|U(y,s)| \\
&\le \frac{2}{x^{k_2}}|U(y,s)|.
\end{aligned}
$$

So, Eq. (48) gives

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty w(x,y)K(x,y)u_1(x,s)|U(y,s)|\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \\
&\le k\int_0^t\int_0^\infty\int_0^\infty \frac{1}{x^{k_2}}\frac{(1+x)^\lambda(1+y)^\lambda}{(xy)^\sigma}u_1(x,s)|U(y,s)|\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \\
&\le k\int_0^t\int_0^\infty\int_0^\infty \frac{1}{x^{k_2}}\frac{(1+x)^\lambda}{(x)^\sigma}u_1(x,s)\,\mathrm{d}x\frac{(1+y)^\lambda}{y^\sigma}|U(y,s)|\,\mathrm{d}y\mathrm{d}s \\
&= k\int_0^t\left[\int_0^\infty \frac{1}{x^{k_2}}\frac{(1+x)^\lambda}{x^\sigma}u_1(x,s)\,\mathrm{d}x\int_0^\infty\frac{(1+y)^\lambda}{y^\sigma}|U(y,s)|\,\mathrm{d}y\right]\mathrm{d}s \\
&\le k\int_0^t M(s)\int_0^\infty \frac{1}{x^{k_2}}\frac{1+x^\lambda}{(x)^\sigma}u_1(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&\le 4k\cdot 6\|u_1\|\int_0^t M(s)\,\mathrm{d}s, \qquad \text{where } c_1 \text{is a constant.} \\
&= \ \Gamma_1\int_0^t M(s)\,\mathrm{d}s, \qquad \text{where } \Gamma_1 = 4k\cdot 6\|u_1\|. \qquad (51)
\end{aligned}
$$

In a similar way, we can show that for the integral (49),

$$\frac{1}{2}\int_0^t\int_0^\infty\int_0^\infty w(x,y)K(x,y)u_2(x,s)U(y,s)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \le \Gamma_2\int_0^t M(s)\,\mathrm{d}s. \tag{52}$$

For the relation (50), we have

$$\begin{aligned}
&\int_0^t\int_0^\infty\left(x+\frac{1}{x^{k_2}}\right)\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,\mathrm{d}y - S(x)|U(x,s)|\right]\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_0^t\int_0^\infty x\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,\mathrm{d}y - S(x)|U(x,s)|\right]\mathrm{d}x\,\mathrm{d}s\\
&+\int_0^t\int_0^\infty \frac{1}{x^{k_2}}\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,\mathrm{d}y - S(x)|U(x,s)|\right]\mathrm{d}x\,\mathrm{d}s
\end{aligned}\tag{53}$$

changing the order of integrations, we get

$$\int_0^t\left[\int_0^\infty\int_0^y xb(x,y)S(y)|U(y,s)|\,\mathrm{d}x\,\mathrm{d}y - \int_0^\infty yS(y)|U(y,s)|\,\mathrm{d}y\right]\mathrm{d}s \tag{54}$$

$$+\int_0^t\int_0^\infty\left[\int_0^y\frac{1}{x^{k_2}}b(x,y)S(y)|U(y,s)|\,\mathrm{d}x - S(y)|U(y,s)|\right]\mathrm{d}y\mathrm{d}s. \tag{55}$$

To Eq. (54), we use (3) and hence the terms in (54) vanishes, and for Eq. (55), we use the condition (iii) of Theorem 2.1 and get

$$\begin{aligned}
&\int_0^t\int_0^\infty\left(x+\frac{1}{x^{k_2}}\right)\left[\int_x^\infty b(x,y)S(y)|U(y,s)|\,\mathrm{d}y - S(x)|U(x,s)|\right]\mathrm{d}x\,\mathrm{d}s\\
&\le (N-1)\int_0^t\int_0^\infty \frac{1}{y^{k_2}}S(y)|U(y,s)|\,\mathrm{d}y\mathrm{d}s\\
&\le (N-1)S_1\int_0^t\int_0^\infty y^{\beta-k_2}|U(y,s)|\,\mathrm{d}y\mathrm{d}s\\
&\le \Gamma_3\int_0^t M(s)\,\mathrm{d}s,\ \text{where}\Gamma_3 = S_1\,(N-1)\,.
\end{aligned}\tag{56}$$

Using the relations (51), (52), and (56) in the Eqs. (48), (49), and (50), we get the following

$$M(t)\le\ \Gamma\int_0^t M(s)\,\mathrm{d}s,\quad \text{where } \Gamma = [\Gamma_1+\Gamma_2+\Gamma_3]\,.$$

Applying Gronwall's Lemma, we get

$$\begin{aligned}
&M(t)\le 0\cdot\exp(\Gamma t)\\
&\Rightarrow M(t) = 0\\
&\Rightarrow \int_0^\infty\left(x+\frac{1}{x^{k_2}}\right)|U(x,t)|\ \mathrm{d}x = 0\\
&\Rightarrow\ u_1(x,t) = u_2(x,t).
\end{aligned}$$

Hence, the uniqueness of the solution has been obtained.

5. Result and discussion

In this work, a complete discussion over the existence and uniqueness theory along with the mass conservation property of the solution for the continuous coagulation fragmentation equation has been done.

The coagulation kernel is assumed to have singularity on both the coordinate axes, and the fragmentation kernel describes the breakage of a particle into multiple fragments. The theory has been discussed by considering the least possible assumptions over both the kernels. In brief, for the existence of the solutions, firstly a suitable truncation of the coagulation kernel and the selection function has been done. This truncation has generated a sequence of functions $f_n(x,t)$. By showing that sequence to be relatively compact, a convergent subsequence has been extracted from them. After that, it has been then proved that the solution to the original problem (1), (2) is actually the limit function of the subsequence, which converges strongly in a subset of $\mathbb{R}^+$. Further, it has been shown that the solution, if exists, satisfies the mass conservation property. The existence theory is able to develop the theoretic foundation for many kernels that have immense practical usage *viz.* Smoluchowski's kernel for Brownian motion, granulation kernels, shear kernels with both linear and nonlinear velocity profile, equipartition of kinetic energy (EKE) kernels, activated sludge flocculation kernel by Ding et al. [5], and the kernel showing aerosol dynamics by Friedlander [10]. The uniqueness of the solution has been showed for a specific class of domains with certain additional restriction over the selection function. But still the uniqueness theory is wide enough to cover all the practically important kernels, mentioned above. So, combining both the theories together it can be said that the existence uniqueness theory besides covering a huge class of coagulation–fragmentation equations, it also includes many of the most well-known and practically important kernels. But this theory is still unable to take the nonrandom coalescence kernels into consideration.

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