

# Adomian decomposition method for solving fragmentation and aggregation population balance equations

Randhir Singh · Jitraj Saha · Jitendra Kumar

Received: 8 February 2014 / Published online: 10 July 2014  
© Korean Society for Computational and Applied Mathematics 2014

**Abstract** In this paper, Adomian decomposition method (ADM) is used to obtain the exact and numerical solutions of fragmentation (breakage) and aggregation (coalescence) population balance equations. The major advantage of the ADM over the traditional numerical discretization methods is that it allows to solve both nonlinear initial and boundary value problems without un-physical restrictive assumptions such as linearization, discretization, perturbation and guessing the initial term or a set of basis functions. It approximates the solution in the form of series with easily computable solution components. Convergence of the series solution is discussed. Convergence analysis is reliable enough to estimate the maximum absolute truncated error of the series solution. Some examples are included to show the accuracy, applicability, and generality of the method.

**Keywords** Fragmentation equations · Aggregation equations · Adomian decomposition method · Population balance equations · Semi-analytical · Approximations

## 1 Introduction

Hulburt [1] and Randolph [2] were the first to propose the framework of population balances in chemical engineering literature to model particulate processes [3]. The

---

R. Singh (✉) · J. Saha · J. Kumar  
Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India  
e-mail: randhir.math@gmail.com, randhir.math@maths.iitkgp.ernet.in

J. Saha  
e-mail: jitrajsaha@yahoo.com

J. Kumar  
e-mail: jkumar@maths.iitkgp.ernet.in

process by which two or more particles undergo changes in its physical properties like mass or volume is called the particulate process. Before we proceed to define our problem, we first give an impression on the significant applications of the particulate processes in real life. Particulate processes are well known in various branches of engineering including crystallization, precipitation, polymerization and various particles related engineering problems. Its applications can be found in many areas including chemistry (reacting polymers), physics (aggregation of colloidal particles, growth of gas bubbles in solids), astrophysics (formation of stars and planets) and meteorology (merging of drops in atmospheric clouds).

The aim of this paper is to use the Adomian decomposition method (ADM) to solve the binary fragmentation (breakage) population balance Eq. [4,5] of the following type

$$\frac{\partial u(t, x)}{\partial t} = \int_x^\infty b(x, y)u(t, y)s(y)dy - s(x)u(t, x),$$

$$u(t, x) \in J = \{[0, T] \times [0, \infty)\}, \quad (1.1)$$

with the initial condition

$$u(0, x) = u_0(x) \geq 0. \quad (1.2)$$

Here  $b(x, y)$  is the breakage function for the formation of particles of size  $x$  from particle of size  $y$  at time  $t$ . Selection function  $s(y)$  describes the rate at which particles are selected to break. In a breakage process, particles break into two or many fragments. Breakage has a significant effect on the number of particles. The total number of particles in a breakage process increases while the total mass remains constant.

In addition, we also consider the binary aggregation (coalescence) population balance equation of the form [6–12]

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \int_0^x a(x-y, y)u(t, x-y)u(t, y)dy - \int_0^\infty a(x, y)u(t, x)u(t, y)dy,$$

$$(1.3)$$

with the initial condition

$$u(0, x) = u_0(x) \geq 0. \quad (1.4)$$

Here  $u(t, x) \in J$  and  $u(t, x)$  represents the concentration of particles of size  $x$  at time  $t$ . Here,  $a(x, y)$  is the aggregation kernel, which describes the rate at which the particles of sizes  $x$  and  $y$  coagulate to form a particle of size  $x + y$ . It is non-negative and symmetric. The first term on the right hand side of (1.3) gives the rate of change of particles of size  $x$  due to aggregation of particles of size  $x - y$  and  $y$ . The second term represents the depletion of particles of size  $x$  by particles coagulating with particles of other size.

The fragmentation population balance equations (1.1) has attracted a lot of attention of many authors [4, 5, 13–19] and many of the references therein. Since (1.1) is a partial integro-differential equation and its numerical solution requires special techniques. This may be the main reason for the development of several methods for its solution, obtained from various scientific disciplines. In [4, 16–18] authors found the exact solution of (1.1) for very simple forms of the breakage kernels and selection functions.

Many different techniques have been used in [7–10, 20, 21] to solve the aggregation population balance equation (1.3). In [7] authors used the numerical scheme based on a conservative formulation and a finite volume approach to solve (1.3). Ranjbar et al. [9] used the Taylor polynomials and radial basis functions together to solve (1.3) with constant kernel. The Laplace-variational iteration method was carried out in [8] to obtain approximate series solution of (1.3). Recently, the homotopy perturbation method [10, 20] and homotopy analysis method [21] were also applied to solve (1.3).

In this paper we will use the ADM for solving the fragmentation and aggregation population balance equations. Recently, the ADM has been used to solve the various scientific models by researchers [22–36]. Adomian [36] asserted that the ADM provides an efficient and computationally suitable method for generating approximate series solution for differential, integral and partial integro-differential equations. Furthermore, the ADM allows solution of both linear and nonlinear functional equations and provides an accurate analytic approximation of the problems. It is well known that the ADM can be applied directly in a straightforward manner without using restrictive assumptions or linearization and discretization of variable. Unlike other numerical or discrete methods, the ADM does not result in any large system of linear or nonlinear equations as it does not require any linearization or perturbation. Thus, it is not much affected by computational round-off errors and there is no requirement of large computer memory and time, compared with the other methods.

The organizations of this paper is as follows. In Sect. 2, the ADM is used to solve (1.1) with initial condition (1.2). In Sect. 2, we also discuss the convergence and maximum truncation error of the series solution obtained by the ADM of the problem. Section 3 deals with the ADM to solve (1.3)–(1.4) and the convergence and the maximum truncation error of series solution. In Sect. 4, the reliability and efficiency of the proposed methods are demonstrated by several numerical examples.

## 2 The ADM for fragmentation equations

In this section, we use the ADM for solving (1.1) analytically. According to the ADM, we first rewrite (1.1) in an operator form as follows:

$$\mathcal{L}u(t, x) = \int_x^\infty b(x, y)u(t, y)s(y)dy - s(x)u(t, x). \quad (2.1)$$

Here,  $\mathcal{L} = \frac{\partial}{\partial t}$  is linear partial differential operator. It is assumed that the solution of the problem (1.1) exists and unique. The operator  $\mathcal{L}^{-1}$  regarded as the inverse operator of  $\mathcal{L}$ , is defined as

$$\mathcal{L}^{-1}[\cdot] := \int_0^t [\cdot] dt. \tag{2.2}$$

Operating with  $\mathcal{L}^{-1}[\cdot]$  on both sides of (2.1) and using the condition  $u(0, x) = u_0(x)$ , we obtain

$$u(t, x) = u_0(x) + \mathcal{L}^{-1} \left[ \int_x^\infty b(x, y) u(t, y) s(y) dy - s(x) u(t, x) \right]. \tag{2.3}$$

The idea of the ADM relies on decomposing the solution  $u(t, x)$  into an infinite series as

$$u(t, x) = \sum_{j=0}^\infty u_j(t, x) \text{ and } u(t, y) = \sum_{j=0}^\infty u_j(t, y). \tag{2.4}$$

Substituting (2.4) into (2.3), we have

$$\begin{aligned} \sum_{j=0}^\infty u_j(t, x) &= u_0(x) + \mathcal{L}^{-1} \times \left[ \int_x^\infty b(x, y) s(y) \left( \sum_{j=0}^\infty u_j(t, y) \right) dy \right. \\ &\quad \left. - s(x) \left( \sum_{j=0}^\infty u_j(t, x) \right) \right]. \end{aligned} \tag{2.5}$$

Comparing both sides of (2.5), we have the recursive scheme as

$$\left. \begin{aligned} u_0(t, x) &= u_0(x), \\ u_j(t, x) &= \mathcal{L}^{-1} \left[ \int_x^\infty b(x, y) u_{j-1}(t, y) s(y) dy - s(x) u_{j-1}(t, x) \right], \quad j = 1, 2, \dots, \end{aligned} \right\} \tag{2.6}$$

which leads a complete determination of the solution components  $u_j(t, x)$ . Hence, the  $n$ -term truncated series solution can be obtained as

$$\psi_n(t, x) = \sum_{j=0}^n u_j(t, x). \tag{2.7}$$

### 2.1 Convergence analysis

In this section we follow the approach discussed in [23] for convergence analysis and error estimation of the proposed recursive scheme (2.6). Convergence analysis is reliable enough to estimate the maximum absolute truncated error of the series

solution. For that reason, let  $\mathbb{X} = ([0, T] \times L^1[0, \infty), \|\cdot\|)$ , where be a Banach space (see, [38]) with the norm defined as

$$\|u\| = \sup_{t \in [0, t_0]} \int_0^\infty \exp(\lambda x) |u(t, x)| dx, \text{ with } \lambda > 0. \tag{2.8}$$

We now rewrite (2.3) in an operator form as follows

$$u = \mathcal{T}u, \tag{2.9}$$

where  $\mathcal{T} : \mathbb{X} \rightarrow \mathbb{X}$  is a linear operator given by

$$\mathcal{T}u = u_0(x) + \mathcal{L}^{-1} \left[ \int_x^\infty b(x, y)u(t, y)s(y)dy - s(x)u(t, x) \right]. \tag{2.10}$$

To show the operator  $\mathcal{T}$  to be contractive, we rewrite the relation (2.10) in the following equivalent form

$$\frac{\partial}{\partial t} [u(t, x) \exp[A(t, x)]] = \exp[A(t, x)] \left[ \int_x^\infty s(y)b(x, y)u(t, y) dy \right],$$

where  $\exp[A(t, x)] = ts(x)$ . Thus we have

$$\begin{aligned} \tilde{\mathcal{T}}u &= u_0(x) \exp[-A(t, x)] \\ &+ \int_0^t \exp[A(x, s) - A(t, x)] \int_x^\infty s(y)b(x, y)u(y, s) dy ds. \end{aligned} \tag{2.11}$$

Since  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$  are equivalent, so it is enough to show  $\tilde{\mathcal{T}}$  is contractive.

**Theorem 2.1** *Let the linear operator  $\tilde{\mathcal{T}}$  defined by (2.11) be contractive, that is,  $\|\tilde{\mathcal{T}}u - \tilde{\mathcal{T}}u^*\| \leq \delta \|u - u^*\|$ , for all  $u, u^* \in \mathbb{X}$  with*

1.  $b(x, y) = c \frac{x^{r-1}}{y^r}$  where  $r = 1, 2, \dots$  and  $c > 0$  is a constant satisfying  $\int_0^y xb(x, y) dx = y$ ,
2.  $s(x) \leq x^k$ , where  $k = 1, 2, \dots$ ,
3.  $\lambda$  is so chosen that  $[\exp(\lambda y) - 1] < 1$  and,
4.  $\delta := \frac{(k!)t_0}{\lambda^{k+1}} c < 1$  for some suitable  $t_0$ .

*Proof* For any  $u$  and  $u^*$ , consider

$$\|\tilde{\mathcal{T}}u - \tilde{\mathcal{T}}u^*\| \leq \int_0^t \left[ \int_0^\infty \exp \left[ \lambda x - (t-s)s(x) \right] \int_x^\infty s(y)b(x, y) |u - u^*|(y, s) dy dx \right] ds$$

$$\begin{aligned}
 &\leq \int_0^t \left[ \int_0^\infty \exp(\lambda x) \int_x^\infty s(y)b(x, y) |u - u^*|(y, s) dy dx \right] ds \quad [ : s \leq t ] \\
 &\leq c \int_0^t \left[ \int_0^\infty \left( \int_0^y \exp(\lambda x) \frac{x^{r-1}}{y^r} dx \right) s(y) |u - u^*|(y, s) dy \right] ds \\
 &\leq c \int_0^t \left[ \int_0^\infty \frac{1}{\lambda y} [\exp(\lambda y) - 1] s(y) |u - u^*|(y, s) dy \right] ds \quad [ : x \leq y ] \\
 &\leq \frac{1}{\lambda} c \int_0^t \left[ \int_0^\infty y^k |u - u^*|(y, s) dy \right] ds \\
 &\leq \frac{k!}{\lambda^{k+1}} c \int_0^t \left[ \int_0^\infty \exp(\lambda y) |u - u^*|(y, s) dy \right] ds \\
 &\leq \frac{k!}{\lambda^{k+1}} c \int_0^t \|u - u^*\| ds \leq \frac{(k!)t_0}{\lambda^{k+1}} c \|u - u^*\| = \delta \|u - u^*\|.
 \end{aligned}$$

If  $\delta = \frac{(k!)t_0}{\lambda^{k+1}} c < 1$  then  $\tilde{T}$  is a contractive mapping. □

**Theorem 2.2** Assume that all the conditions of Theorem 3.1 hold. Let  $u_0, u_1, u_2, \dots$ , be the components of the solution  $u$  given by the recursive scheme (2.6), and let  $\psi_n = \sum_{j=0}^n y_j$  be the  $n$ -terms series solution defined by (2.7). Then the series solution  $\psi_n$  converges whenever  $\delta = \frac{(k!)t_0}{\lambda^{k+1}} c < 1$  and  $\|u_1\| < \infty$ .

*Proof* From (2.6) and (2.7), we have

$$\begin{aligned}
 \psi_n &= \sum_{j=0}^n u_j(t, x) = u_0(x) + \sum_{j=1}^n \mathcal{L}^{-1} \left[ \int_x^\infty b(x, y)s(y)u_{j-1}(t, y)dy - s(x)u_{j-1}(t, x) \right]. \\
 &= u_0(x) + \mathcal{L}^{-1} \left[ \int_x^\infty b(x, y)s(y) \left( \sum_{j=0}^{n-1} u_j(t, y) \right) dy - s(x) \left( \sum_{j=0}^{n-1} u_j(t, x) \right) \right]. \\
 &= u_0(x) + \mathcal{L}^{-1} \left[ \int_x^\infty b(x, y)s(y)\psi_{n-1}(t, y)dy - s(x)\psi_{n-1}(t, x) \right],
 \end{aligned}$$

which is equivalent to the following operator equation form as

$$\psi_n = \mathcal{T}\psi_{n-1}. \tag{2.12}$$

By following the steps of Theorem 2.12, we obtain

$$\|\psi_{m+1} - \psi_m\| \leq \delta \|\psi_m - \psi_{m-1}\|.$$

Thus we have

$$\|\psi_{m+1} - \psi_m\| \leq \delta \|\psi_m - \psi_{m-1}\| \leq \delta^2 \|\psi_{m-1} - \psi_{m-2}\| \leq \dots \leq \delta^m \|\psi_1 - \psi_0\|.$$

Using the triangle inequality with  $n > m$  we have

$$\begin{aligned} \|\psi_n - \psi_m\| &\leq \|\psi_n - \psi_{n-1}\| + \|\psi_{n-1} - \psi_{n-2}\| + \dots + \|\psi_{m+1} - \psi_m\|, \\ &\leq [\delta^{n-1} + \delta^{n-2} + \dots + \delta^m] \|\psi_1 - \psi_0\|, \\ &= \delta^m [1 + \delta + \delta^2 + \dots + \delta^{n-m-1}] \|\psi_1 - \psi_0\| = \delta^m \left( \frac{1 - \delta^{n-m}}{1 - \delta} \right) \|u_1\|. \end{aligned}$$

Since  $0 < \delta < 1$  so,  $(1 - \delta^{n-m}) < 1$ , and  $\|u_1\| < \infty$ . It follows that

$$\|\psi_n - \psi_m\| \leq \frac{\delta^m}{1 - \delta} \|u_1\|, \tag{2.13}$$

which converges to zero as  $m \rightarrow \infty$ . This implies that there exists a  $\psi$  such that  $\lim_{n \rightarrow \infty} \psi_n = \psi$ . Since, we have  $u = \sum_{j=0}^{\infty} u_j = \lim_{n \rightarrow \infty} \psi_n = \psi$  which is the exact solution of (2.9).  $\square$

**Theorem 2.3** *Let  $u$  be the exact solution of (2.9) and  $\psi_m$  be the series solution given by (2.7). Then there holds*

$$\|u - \psi_m\| \leq \frac{\delta^m}{(1 - \delta)} \|u_1\|, \tag{2.14}$$

$$\|u_1\| = \sup_{t \in [0, t_0]} \int_0^{\infty} \exp(\lambda x) |u_1(t, x)| dx.$$

*Proof* From the estimate (2.13), for  $n \geq m$ ,  $n, m \in \mathbb{N}$ , we get

$$\|\psi_n - \psi_m\| \leq \frac{\delta^m}{1 - \delta} \|u_1\|.$$

Fixing  $m$  and letting  $n \rightarrow \infty$ , and using  $\lim_{n \rightarrow \infty} \psi_n = u$ , we obtain the desired result of theorem.  $\square$

### 3 The ADM for aggregation equations

In this section, the ADM is used to solve (1.3)–(1.4) approximately. According to the ADM, we rewrite (1.3) in an operator form as follows

$$\mathcal{L}u(t, x) = \frac{1}{2} \int_0^x a(x - y, y) f_1(u) dy - \int_0^{\infty} a(x, y) f_2(u) dy. \tag{3.1}$$

Here,  $\mathcal{L} = \frac{\partial}{\partial t}$  is a linear partial differential operator. The nonlinear functions are denoted by  $f_1(u) = u(t, x - y)u(t, y)$  and  $f_2(u) = u(t, x)u(t, y)$ .

Operating with  $\mathcal{L}^{-1}[\cdot]$  defined by (2.2) on both sides of (3.1) and using  $u(0, x) = u_0(x)$  yields

$$u(t, x) = u_0(x) + \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x a(x - y, y) f_1(u) dy - \int_0^\infty a(x, y) f_2(u) dy \right]. \tag{3.2}$$

The ADM introduces the solution  $u(t, x)$  and the nonlinear functions  $f_1(u)$  and  $f_2(u)$  by infinite series

$$u(t, x) = \sum_{j=0}^\infty u_j(t, x) \tag{3.3}$$

and

$$f_1(u) = \sum_{j=0}^\infty A_j, \quad f_2(u) = \sum_{j=0}^\infty B_j, \tag{3.4}$$

where  $A_j$  and  $B_j$  are the so-called Adomian polynomials and defined by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ f_1 \left( \sum_{k=0}^\infty u_k \lambda^k \right) \right]_{\lambda=0}, \quad B_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ f_2 \left( \sum_{k=0}^\infty u_k \lambda^k \right) \right]_{\lambda=0},$$

as given in [35]. Recently, [28–32, 37] authors developed several new efficient algorithms for rapid computer-generation of the Adomian polynomials.

Substituting (3.3) and (3.4) into (3.2) yields

$$\sum_{j=0}^\infty u_j(t, x) = u_0(x) + \mathcal{L}^{-1} \times \left[ \frac{1}{2} \int_0^x a(x - y, y) \left( \sum_{j=0}^\infty A_j \right) dy - \int_0^\infty a(x, y) \left( \sum_{j=0}^\infty B_j \right) dy \right]. \tag{3.5}$$

Comparing both sides of (3.5), we have

$$\left. \begin{aligned} u_0(t, x) &= u_0(x), \\ u_j(t, x) &= \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x a(x - y, y) A_{j-1} dy - \int_0^\infty a(x, y) B_{j-1} dy \right], \quad j = 1, 2, \dots, \end{aligned} \right\} \tag{3.6}$$

which gives the complete determination of the solution components  $u_j(t, x)$ . Hence, the  $n$ -terms truncated series solution can be obtained as

$$\psi_n(t, x) = \sum_{j=0}^n u_j(t, x). \tag{3.7}$$



**Note 3.1** Note that both the recursive schemes (2.6) and (3.6) can be used to get approximate/exact solutions of the fragmentation and aggregation population balance equations provided the integrals appear in (2.6) and (3.6) to be evaluated exactly.

### 3.1 Convergence analysis

We now discuss the convergence of the recursive scheme (3.6). Let  $\mathbb{X} = (\mathcal{C}([0, T] : L^1[0, \infty)), \|\cdot\|)$  be a Banach space (see, [38]) with the norm defined as

$$\|u\| = \sup_{s \in [0, t_0]} \int_0^\infty |u(s, x)| dx < \infty. \tag{3.8}$$

We write (3.2) in an operator equation form as

$$u = \mathcal{N}u, \tag{3.9}$$

where  $\mathcal{N} : \mathbb{X} \rightarrow \mathbb{X}$  is a nonlinear operator given by

$$\mathcal{N}u = u_0(x) + \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x a(x-y, y) f_1(u) dy - \int_0^\infty a(x, y) f_2(u) dy \right]. \tag{3.10}$$

To show  $\mathcal{N}$  is contractive, we write the Eq. (3.10) in the following equivalent form

$$\frac{\partial}{\partial t} [u(t, x) \exp[A(x, t, u)]] = \frac{1}{2} \exp[A(x, t, u)] \int_0^x a(x-y, y) u(t, x-y) u(t, y) dy, \tag{3.11}$$

where  $A(x, t, u) = \int_0^t \int_0^\infty a(x, y) u(s, y) dy ds$ . Thus we have

$$\begin{aligned} \tilde{\mathcal{N}}u &= u_0(x) \exp[-A(x, t, u)] \\ &+ \frac{1}{2} \int_0^t \exp[A(x, s, u) - A(x, t, u)] \int_0^x a(x-y, y) u(s, x-y) u(s, y) dy ds. \end{aligned} \tag{3.12}$$

Since  $\mathcal{N}$  and  $\tilde{\mathcal{N}}$  are equivalent, so it is enough to show  $\tilde{\mathcal{N}}$  is contractive.

**Theorem 3.1** *Let the non-linear operator  $\tilde{\mathcal{N}}$  defined by (3.12) be contractive, that is,  $\|\tilde{\mathcal{N}}u - \tilde{\mathcal{N}}u^*\| \leq \delta \|u - u^*\|$ , for all  $u, u^* \in \mathbb{X}$  with*

1.  $a(x, y) = 1$  for all  $x, y \in (0, \infty)$ ,
2.  $\delta := t_0 \exp(2t_0L) [\|u_0\| + 2t_0L^2 + 2t_0L] < 1$ , where  $L = \|u_0\|(T + 1)$ .

*Proof* Let  $u, u^* \in \mathbb{X}$ , consider

$$\begin{aligned} \tilde{\mathcal{N}}u - \tilde{\mathcal{N}}u^* &= u_0(x)H(x, 0, t) + \frac{1}{2} \int_0^t H(x, s, t) \int_0^x u(s, x - y)u(s, y) dy ds \\ &\quad - \frac{1}{2} \int_0^t \exp[A(x, s, u^*) - A(x, t, u^*)] \\ &\quad \cdot \left[ \int_0^x u^*(x - y, s) \{u(y, s) - u^*(y, s)\} dy \right. \\ &\quad \left. + \int_0^x u(y, s) \{u(x - y, s) - u^*(x - y, s)\} dy \right] ds, \end{aligned} \tag{3.13}$$

where  $H(x, s, t) = \exp[A(x, s, u) - A(x, t, u)] - \exp[A(x, s, u^*) - A(x, t, u^*)]$ . It can be shown that,

$$\begin{aligned} |H(x, s, t)| &\leq \exp \left\{ - \int_s^t \int_0^\infty u^*(\tau, y) dy d\tau \right\} (t - s) \|u - u^*\| \\ &\leq (t - s) \exp \{ (t - s)B \} \|u - u^*\| \leq L_1 \|u - u^*\|, \end{aligned} \tag{3.14}$$

where  $L_1 = t \exp \{ tB \}$  and  $B = \max \{ \|u\|, \|u^*\| \}$ . In order to show that the operator  $\tilde{\mathcal{N}}$  is a contractive, let us define the set  $D = \{u \in \mathbb{X} : \|u\| \leq 2L\}$ . It can be shown that the operator  $\tilde{\mathcal{N}}$  maps  $D$  into itself. For  $u, u^* \in D$  we have  $B \leq 2L$ . Taking norm on both sides of (3.13), we get

$$\begin{aligned} \|\tilde{\mathcal{N}}u - \tilde{\mathcal{N}}u^*\| &\leq L_1 \|u_0\| \|u - u^*\| + L_1 \|u - u^*\| \int_0^t \left[ \frac{1}{2} \|u\|^2 \right] ds \\ &\quad + \int_0^t L_1 \left[ \frac{1}{2} (\|u\| + \|u^*\|) \|u - u^*\| \right] ds \\ &\leq L_1 \left[ \|u_0\| + \frac{1}{2} t \|u\|^2 + \frac{1}{2} t (\|u\| + \|u^*\|) \right] \|u - u^*\| \\ &\leq \delta \|u - u^*\|, \end{aligned} \tag{3.15}$$

if  $\delta = t_0 \exp(2t_0L) [\|u_0\| + 2t_0L^2 + 2t_0L] < 1$  under suitably chosen  $t_0$  the operator  $\tilde{\mathcal{N}}$  is a contraction map. □

**Theorem 3.2** *Assume that all the conditions of Theorem 3.1 hold. Let  $u_0, u_1, u_2, \dots$ , be the components of the solution  $u$  given by the recursive scheme (3.6), and let  $\psi_n = \sum_{j=0}^n y_j$  be the  $n$ -terms series solution defined by (3.7). Then the series solution  $\psi_n$  converges whenever  $\delta = t_0 \exp(2t_0L) [\|u_0\| + 2t_0L^2 + 2t_0L] < 1$  and  $\|u_1\| < \infty$ .*

*Proof* Using (3.6) and (3.7), we have

$$\begin{aligned} \psi_n &= \sum_{j=0}^n u_j(t, x) = u_0(x) + \sum_{j=1}^n \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x a(x-y, y) A_{j-1} dy - \int_0^\infty a(x, y) B_{j-1} dy \right], \\ &= u_0(x) + \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x a(x-y, y) \left( \sum_{j=0}^{n-1} A_j \right) dy - \int_0^\infty a(x, y) \left( \sum_{j=0}^{n-1} B_j \right) dy \right]. \end{aligned} \tag{3.16}$$

As we know  $\sum_{j=0}^n A_j \leq f_1(\psi_n)$  and  $\sum_{j=0}^n B_j \leq f_2(\psi_n)$  as given in ([32] pp. 945) and using it in (3.16), we get

$$\psi_n \leq u_0(x) + \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x a(x-y, y) f_1(\psi_{n-1}) dy - \int_0^\infty a(x, y) f_2(\psi_{n-1}) dy \right] \tag{3.17}$$

which is equivalent to the following operator form

$$\psi_n \leq \mathcal{N}\psi_{n-1}. \tag{3.18}$$

By following the steps of Theorem 3.1, we obtain

$$\|\psi_{n+1} - \psi_n\| \leq \delta \|\psi_n - \psi_{n-1}\|.$$

Thus, we have

$$\|\psi_{n+1} - \psi_n\| \leq \delta \|\psi_m - \psi_{n-1}\| \leq \delta^2 \|\psi_{n-1} - \psi_{n-2}\| \leq \dots \leq \delta^n \|\psi_1 - \psi_0\|.$$

Using the triangle inequality for all  $n, m \in \mathbb{N}$  with  $n > m$ , we have

$$\begin{aligned} \|\psi_n - \psi_m\| &\leq \|\psi_n - \psi_{n-1}\| + \|\psi_{n-1} - \psi_{n-2}\| + \dots + \|\psi_{m+1} - \psi_m\|, \\ &\leq (\delta^{n-1} + \delta^{n-2} + \dots + \delta^m) \|\psi_1 - \psi_0\| = \delta^m \left( \frac{1 - \delta^{n-m}}{1 - \delta} \right) \|u_1\|. \end{aligned}$$

As  $\delta < 1$  so,  $(1 - \delta^{n-m}) < 1$ , and  $\|u_1\| < \infty$ , it follows that

$$\|\psi_n - \psi_m\| \leq \frac{\delta^m}{1 - \delta} \|u_1\|, \tag{3.19}$$

which converges to zero as  $m \rightarrow \infty$ . This implies that there exists a  $\psi$  such that  $\lim_{n \rightarrow \infty} \psi_n = \psi$ . Since, we have  $u = \sum_{j=0}^\infty u_j = \lim_{n \rightarrow \infty} \psi_n = \psi$ , which is the exact solution of (3.9).  $\square$

**Theorem 3.3** *Let  $u$  be the exact solution of (3.9) and  $\psi_m$  be the series solution defined by (3.7). Then there holds*

$$\|u - \psi_m\| \leq \frac{\delta^m}{(1 - \delta)} \|u_1\|, \tag{3.20}$$

where  $\|u_1\| = \sup_{t \in [0, t_0]} \int_0^\infty |u_1(t, x)| dx$ .

*Proof* The proof is similar to that of Theorem 2.3, hence it is omitted here. □

### 4 Numerical results and discussion

In this section, we will demonstrate the efficiency and accuracy of the ADM with several test problems. All symbolic and numerical calculations are done using the ‘MATHEMATICA SOFTWARE’ Package.

#### 4.1 Example of fragmentation equation

*Example 4.1* Consider (1.1)–(1.2) with breakage kernel  $b(x, y) = \frac{2}{y}$ , selection function  $s(x) = x$  and the initial condition  $u_0(x) = e^{-x}$ , its exact solution is  $u(t, x) = (1 + t)^2 e^{-x(1+t)}$  as in [17, 39].

According to the ADM (2.6), we have the recursive scheme as follows:

$$u_j(t, x) = \mathcal{L}^{-1} \left[ \begin{array}{l} u_0(t, x) = e^{-x}, \\ \int_x^\infty \frac{2}{y} u_{j-1}(t, y) y dy - x u_{j-1}(t, x) \end{array} \right], \quad j = 1, 2, \dots \quad (4.1)$$

Using the recursive scheme (4.1), we obtain the components  $u_j(t, x)$  as follows:

$$\begin{aligned} u_0(t, x) &= e^{-x}, \\ u_1(t, x) &= \frac{(-t)^1 x^{-1}}{1!} (0 - 2x + x^2) e^{-x}, \\ u_2(t, x) &= \frac{(-t)^2 x^0}{2!} (2 - 4x + x^2) e^{-x}, \\ &\vdots \\ u_j(t, x) &= \frac{(-t)^j x^{j-2}}{j!} (j(j-1) - 2jx + x^2) e^{-x}. \end{aligned}$$

Hence, the  $n$ -terms truncated series solution is obtained as  $\psi_n(t, x) = \sum_{j=0}^n u_j(t, x)$ ,

$$\psi_n(t, x) = \sum_{j=0}^n \frac{(-t)^j x^{j-2}}{j!} (j(j-1) - 2jx + x^2) e^{-x}. \quad (4.2)$$

Then by taking the limit of (4.2), we obtain

$$\lim_{n \rightarrow \infty} \psi_n(t, x) = \sum_{j=0}^\infty \frac{(-t)^j x^{j-2}}{j!} (j(j-1) - 2jx + x^2) e^{-x} = (1 + t)^2 e^{-x(1+t)},$$

**Table 1** Truncation error of example 4.1

$m$	5	10	15	20	25	30
$\Delta_m$	0.066339	0.0391725	0.023131	0.0136586	0.00806527	0.00476246

which is the exact solution of Example 4.1.

Table 1 we give the truncation error using the formula given in Theorem 2.3 as

$$\Delta_m := \frac{\delta^m}{(1 - \delta)} \|u_1\|, \quad \|u_1\| = \sup_{t \in [0, t_0]} \int_0^\infty t e^{\lambda x - x} (2 - x) dx = \frac{t_0(1 - 2\lambda)}{(\lambda - 1)^2}$$

Here  $\delta = \frac{k!t_0}{\lambda^{k+1}}c$ ;  $k = 1, c = 2$  are given; and choosing  $\lambda = 0.1; t_0 = 0.0045$ .

*Example 4.2* Consider (1.1)–(1.2) with  $b(x, y) = \frac{2}{y}, s(x) = x^2, u_0(x) = e^{-x}$ , its exact solution is  $u(t, x) = (1 + 2t + 2tx)e^{-x(1+xt)}$  as in [17, 39].

According to the ADM (2.6), we have the following recursive scheme as:

$$u_j(t, x) = \mathcal{L}^{-1} \left[ \int_x^\infty \frac{2}{y} u_{j-1}(t, y) y^2 dy - x^2 u_{j-1}(t, x) \right], \quad j = 1, 2, \dots, \left. \begin{array}{l} u_0(t, x) = e^{-x}, \end{array} \right\} \quad (4.3)$$

Using the recurrence relation (4.3), we obtain the components as follow:

$$\begin{aligned} u_0(t, x) &= e^{-x} \\ u_1(t, x) &= \frac{(-t)^1 x^0}{1!} (-2 - 2x + x^2) e^{-x} \\ u_2(t, x) &= \frac{(-t)^2 x^2}{2!} (-4 - 4x + x^2) e^{-x} \\ &\vdots \\ u_j(t, x) &= \frac{(-t)^j x^{2j-2}}{j!} (-2j - 2jx + x^2) e^{-x}. \end{aligned}$$

Hence, the  $n$ -term truncated series solution can be obtained as

$$\psi_n(t, x) = \sum_{j=0}^n \frac{(-t)^j x^{2j-2}}{j!} (-2j - 2jx + x^2) e^{-x}. \quad (4.4)$$

Taking limit of (4.4), we get

$$\lim_{n \rightarrow \infty} \psi_n(t, x) = \sum_{j=0}^\infty \frac{(-t)^j x^{2j-2}}{j!} (-2j - 2jx + x^2) e^{-x} = (1 + 2t + 2tx) e^{-x(1+xt)},$$

which is the exact solution of Example 4.2.

*Example 4.3* Consider (1.1)–(1.2) with  $b(x, y) = \frac{2}{y}$ ,  $s(x) = x$ , and  $u_0(x) = \delta(x - a)$ , its exact solution is  $u(t, x) = e^{-tx} [\delta(x - a) + (2t + t^2(a - x))\theta(a - x)]$  as in [17, 39].

According to the ADM (2.6), we have

$$u_j(t, x) = \mathcal{L}^{-1} \left[ \int_x^\infty \frac{2}{y} u_{j-1}(t, y) y dy - x u_{j-1}(t, x) \right], \quad j = 1, 2, \dots, \quad (4.5)$$

Using the relation (4.5), the solution components  $u_j(t, x)$  are calculated as

$$\begin{aligned} u_0(t, x) &= \delta(x - a), \\ u_1(t, x) &= \frac{(-xt)^1}{1!} \delta(x - a) + 2t\theta(a - x), \\ u_2(t, x) &= \frac{(-xt)^2}{2!} \delta(x - a) + \frac{(-xt)^1}{1!} 2t\theta(a - x) + t^2(a - x)\theta(a - x), \\ u_3(t, x) &= \frac{(-xt)^3}{3!} \delta(x - a) + \frac{(-xt)^2}{2!} 2t\theta(a - x) + \frac{(-xt)^1}{1!} t^2(a - x)\theta(a - x), \\ &\vdots \\ u_j(t, x) &= \left[ \frac{(-tx)^j}{j!} \delta(x - a) + \frac{(-tx)^{j-1}}{(j - 1)!} 2t\theta(a - x) + \frac{(-tx)^{j-2}}{(j - 2)!} t^2(a - x)\theta(a - x) \right]. \end{aligned}$$

Here  $\delta(x - a)$  is Dirac’s delta function and  $\theta(a - x)$  is unite step function. Hence, the  $n$ -terms truncated series solution is obtained as

$$\begin{aligned} \psi_n(t, x) &= \delta(x - a) \sum_{j=0}^n \frac{(-tx)^j}{j!} + 2t\theta(a - x) \sum_{j=0}^{n-1} \frac{(-tx)^j}{j!} + t^2(a - x)\theta(a - x) \\ &\quad \times \sum_{j=0}^{n-2} \frac{(-tx)^j}{j!}. \end{aligned}$$

By taking limit, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi_n(t, x) &= \delta(x - a) \sum_{j=0}^\infty \frac{(-tx)^j}{j!} + 2t\theta(a - x) \sum_{j=0}^\infty \frac{(-tx)^j}{j!} + t^2(a - x)\theta(a - x) \\ &\quad \times \sum_{j=0}^\infty \frac{(-tx)^j}{j!} = e^{-tx} [\delta(x - a) + (2t + t^2(a - x)) \theta(a - x)], \end{aligned}$$

which is the exact solution of Example 4.3.

*Example 4.4* Consider (1.1)–(1.2) with  $b(x, y) = \frac{2}{y}$ ,  $s(x) = x^2$ , and  $u_0(x) = \delta(x - a)$ . Its exact solution is  $u(t, x) = e^{-tx^2} [\delta(x - a) + 2ta\theta(a - x)]$  as in [17,39].

According to the ADM (2.6), we have the following recursive scheme

$$u_j(t, x) = \mathcal{L}^{-1} \left[ \begin{aligned} &u_0(t, x) = \delta(x - a), \\ &\int_x^\infty \frac{2}{y} u_{j-1}(t, y) y^2 dy - x^2 u_{j-1}(t, x) \end{aligned} \right], j = 1, 2, \dots, \quad (4.6)$$

Using the recursive scheme (4.6), the solution components are obtained as follow

$$\begin{aligned} u_0(t, x) &= \delta(x - a), \\ u_1(t, x) &= \frac{(-tx^2)^1}{1!} \delta(x - a) + 2at\theta(a - x), \\ u_2(t, x) &= \frac{(-tx^2)^2}{2!} \delta(x - a) + \frac{(-tx^2)^1}{1!} 2at\theta(a - x), \\ u_3(t, x) &= \frac{(-tx^2)^3}{3!} \delta(x - a) + \frac{(-tx^2)^2}{2!} 2at\theta(a - x), \\ &\vdots \\ u_j(t, x) &= \left[ \frac{(-tx^2)^j}{j!} \delta(x - a) + \frac{(-tx^2)^{j-1}}{(j-1)!} 2at\theta(a - x) \right]. \end{aligned}$$

Hence, the  $n$ -terms truncated approximate series solution is obtained as follows:

$$\psi_n(t, x) = \delta(x - a) \sum_{j=0}^n \frac{(-tx^2)^j}{j!} + 2ta\theta(a - x) \sum_{j=0}^{n-1} \frac{(-tx^2)^j}{j!}.$$

Hence, taking the limit, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi_n(t, x) &= \delta(x - a) \sum_{j=0}^\infty \frac{(-tx^2)^j}{j!} + 2ta\theta(a - x) \sum_{j=0}^\infty \frac{(-tx^2)^j}{j!}. \\ &= e^{-tx^2} [\delta(x - a) + 2at\theta(a - x)], \end{aligned}$$

which is the exact solution.

*Example 4.5* Consider (1.1)–(1.2) with general selection function  $s(x) = x^k$ , Austin Kernel [40] as

$$b(x, y) = \frac{\left( \frac{\phi\gamma x^{\gamma-1}}{y^\gamma} + \frac{(1-\phi)\lambda x^{\lambda-1}}{y^\lambda} \right)}{\left( \frac{\phi\gamma}{\lambda+1} y + \frac{(1-\phi)\lambda}{\lambda+1} \right)}.$$

and  $u_0(x) = e^{-x}$ . Note that the exact solution of this problem is not available in literature.

According to the ADM (2.6), we have the following recursive scheme

$$\left. \begin{aligned} u_0(t, x) &= e^{-x}, \\ u_j(t, x) &= \mathcal{L}^{-1} \left[ \int_x^\infty b(x, y) u_{j-1}(t, y) s(y) dy - s(x) u_{j-1}(t, x) \right], \quad j = 1, 2, \dots \end{aligned} \right\} \tag{4.7}$$

To obtain the exact solution of this problem, we take some specific values of the parameters  $k, \lambda, \phi$  and  $\gamma$ .

**Case (i) For  $k = 3, \lambda = 2, \phi = 0$  and  $\gamma = 2$ :** We have  $s(x) = x^3$  and  $b(x, y) = \frac{3x}{y^2}$ . Using the recursive scheme (4.7), we obtain the solution components as follows:

$$\begin{aligned} u_0(t, x) &= e^{-x}, \\ u_1(t, x) &= \frac{(-t)^1 x^1}{1!} (-3 - 3x + x^2) e^{-x}, \\ u_2(t, x) &= \frac{(-t)^2 x^4}{2!} (-6 - 6x + x^2) e^{-x}, \\ &\vdots \\ u_j(t, x) &= \frac{(-t)^j x^{3j-2}}{j!} (-3j - 3jx + x^2) e^{-x}. \end{aligned}$$

Hence, the  $n$ -term approximate series solution is obtained as

$$\psi_n(t, x) = \sum_{j=0}^n \left[ \frac{(-t)^j x^{3j-2}}{j!} (-3j - 3jx + x^2) \right] e^{-x}.$$

Taking limit of above equation we get

$$\lim_{n \rightarrow \infty} \psi_n(t, x) = \sum_{j=0}^\infty \left[ \frac{(-t)^j x^{3j-2}}{j!} (-3j - 3jx + x^2) \right] e^{-x} = e^{-x-tx^3} (1 + 3tx + 3tx^2).$$

By Theorem 2.12, if  $\psi_n(t, x)$  converges, then it will converge to the exact solution. Therefore, the limiting value of  $\psi_n(t, x)$  which must be the exact solution.

**Case (ii) For  $k = 4, \lambda = 3, \phi = 0$  and  $\gamma = 3$ :** We have  $s(x) = x^4$  and  $b(x, y) = \frac{4x^2}{y^3}$ . Using (4.7), we obtain the solution components as follows:

$$\begin{aligned} u_0(t, x) &= e^{-x}, \\ u_1(t, x) &= \frac{(-t)^1 x^2}{1!} (-4 - 4x + x^2) e^{-x}, \\ u_2(t, x) &= \frac{(-t)^2 x^6}{2!} (-8 - 8x + x^2) e^{-x}, \\ &\vdots \\ u_j(t, x) &= \frac{(-t)^j x^{4j-2}}{j!} (-4j - 4jx + x^2) e^{-x}. \end{aligned}$$



The  $n$ -term truncated series solution is obtained as

$$\psi_n(t, x) = \sum_{j=0}^n \left[ \frac{(-t)^j x^{4j-2}}{j!} (-4j - 4jx + x^2) \right] e^{-x}.$$

Then we take limit

$$\lim_{n \rightarrow \infty} \psi_n(t, x) = \sum_{j=0}^{\infty} \left[ \frac{(-t)^j x^{4j-2}}{j!} (-4j - 4jx + x^2) \right] e^{-x} = e^{-x-tx^4} (1 + 4tx^2 + 4tx^3).$$

Since the exact solution is not known for this problem therefore by Theorem 2.12, if the sequence  $\psi_n(t, x)$  converges, then it will converge to exact solution. But the limiting value of  $\psi_n(t, x)$  gives  $e^{-x-tx^4} (1 + 4tx^2 + 4tx^3)$  so this must be exact solution of the problem.

**Case (iii) In general  $\lambda = k - 1, \phi = 0$  and  $\gamma = k - 1$ :** We have  $s(x) = x^k$  and  $b(x, y) = \frac{kx^{k-2}}{y^{k-1}}$ . Using (4.6), we obtain the components as

$$\begin{aligned} u_0(t, x) &= e^{-x}, \\ u_1(t, x) &= \left[ \frac{(-t)^1 x^{1k-2}}{1!} (-1k - 1kx + x^2) \right] e^{-x}, \\ u_2(t, x) &= \left[ \frac{(-t)^2 x^{2k-2}}{2!} (-2k - 2kx + x^2) \right] e^{-x}, \\ &\vdots \\ u_j(t, x) &= \left[ \frac{(-t)^j x^{jk-2}}{j!} (-jk - jkx + x^2) \right] e^{-x}. \end{aligned}$$

The  $n$ -term approximate series solution can be obtained as

$$\psi_n(t, x) = \sum_{j=0}^n \left[ \frac{(-t)^j x^{jk-2}}{j!} (-jk - jkx + x^2) \right] e^{-x}.$$

The limiting value of above sequence is given as

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi_n(t, x) &= \sum_{j=0}^{\infty} \left[ \frac{(-t)^j x^{jk-2}}{j!} (-kj - kjx + x^2) \right] e^{-x}. \\ &= e^{-x-tx^k} (1 + ktx^{k-2} + ktx^{k-1}). \end{aligned}$$

Therefore, the limiting value of  $\psi_n(t, x)$  gives  $e^{-x-tx^k} (1 + ktx^{k-2} + ktx^{k-1})$  which must be the exact solution.

*Remark 4.1* Note that in **Case (iii)**, we are able to obtain the exact solution for any  $k \geq 2, k \in \mathbb{N}$ .

**Case (iv) For  $k = 1, \lambda = 2, \phi = 0$  and  $\gamma = 2$ :** We have  $s(x) = x$  and  $b(x, y) = \frac{3x}{y^2}$ . Using (4.6), we have the solution components as

$$\begin{aligned}
 u_0(t, x) &= e^{-x}, \\
 u_1(t, x) &= \frac{-1^2 t x^{-1}}{1!} [0 - 0x - x^2 + 3x^2 e^x \Gamma(0, x)] e^{-x}, \\
 u_2(t, x) &= \frac{-1^3 t^2 x^0}{2!} [0 - 6x - x^2 + 12e^x x^2 \Gamma(0, x)] e^{-x}, \\
 u_3(t, x) &= \frac{-1^4 t^3 x^1}{3!} [3 - 21x - x^2 + 30e^x x^2 \Gamma(0, x)] e^{-x}, \\
 &\vdots \\
 u_j(t, x) &= \frac{(-1)^{j+1} t^j x^{j-2}}{j!} \left[ \frac{j(j-1)(j-2)}{2} - \frac{j(j-1)(j+4)}{2} x - x^2 + \frac{j(j+1)(j+2)}{2} e^x x^2 \Gamma(0, x) \right] e^{-x}.
 \end{aligned}$$

The truncated  $n$ -term approximate series solution is obtained as

$$\begin{aligned}
 \psi_n(t, x) &= \sum_{j=0}^n \frac{(-1)^{j+1} t^j x^{j-2}}{j!} \left[ \frac{j(j-1)(j-2)}{2} - \frac{j(j-1)(j+4)}{2} x - x^2 + \frac{j(j+1)(j+2)}{2} e^x x^2 \Gamma(0, x) \right] e^{-x}.
 \end{aligned}$$

By taking limit, we obtain

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \psi_n(t, x) &= \sum_{j=0}^{\infty} \frac{(-1)^{j+1} t^j x^{j-2}}{j!} \left[ \frac{j(j-1)(j-2)}{2} - \frac{j(j-1)(j+4)}{2} x - x^2 + \frac{j(j+1)(j+2)}{2} e^x x^2 \Gamma(0, x) \right] e^{-x}. \\
 \psi(t, x) &= \frac{1}{2} e^{-x-tx} \left[ 2 + 6t^2x + t^3x - t^3x^2 + 6e^x tx \Gamma(0, x) - 6e^x t^2 x^2 \Gamma(0, x) + e^x t^3 x^3 \Gamma(0, x) \right].
 \end{aligned}$$

The limiting value of  $\psi_n(t, x)$  gives  $\frac{1}{2} e^{-x-tx} [2 + 6t^2x + t^3x - t^3x^2 + 6e^x tx \Gamma(0, x) - 6e^x t^2 x^2 \Gamma(0, x) + e^x t^3 x^3 \Gamma(0, x)]$  which must be the exact solution. Here,  $\Gamma(0, x)$  is incomplete Gamma function and defined as  $\Gamma(0, x) = \int_x^\infty t^{-1} e^{-t} dt$ .

*Remark 4.2* Note that we have shown only those case where we have been able to get the closed of the exact solutions. However, an approximate solution can be obtained for any breakage kernels and selection function with any initial conditions provided the solution of the problem exists.

### 4.2 Example of aggregation equation

*Example 4.6* Consider (1.3)–(1.4) with constant kernel  $a(x, y) = 1$ ,  $u_0(x) = e^{-x}$ . The exact solution is

$$u(t, x) = N^2(t)e^{-N(t)x},$$

where  $N(t) = \frac{2M_0}{2+M_0t}$  and  $M_0 = 1$  as in [9].

According to the ADM (3.6), we have the following recursive scheme as

$$u_j(t, x) = \mathcal{L}^{-1} \left[ \frac{u_0(t, x) = e^{-x}, \right. \\ \left. \frac{1}{2} \int_0^x A_j dy - \int_0^\infty B_j dy \right], j = 1, 2, \dots, \quad (4.8)$$

where  $A_j$  and  $B_j$  are the Adomian polynomials of the nonlinear functions  $f_1(u) = u(t, x - y)u(t, y)$  and  $f_2(u) = u(t, x)u(t, y)$ . Using the definitional formula (3.4), we compute  $A_j$  and  $B_j$  as follows:

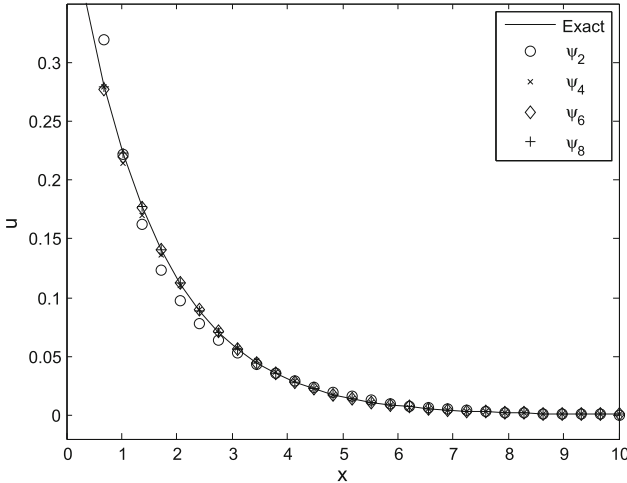
$$A_n = \sum_{j=0}^n u_j(t, x - y)u_{n-j}(t, y) \text{ and } B_n = \sum_{j=0}^n u_j(t, x)u_{n-j}(t, y). \quad (4.9)$$

In view of (4.8) and (4.9), we obtain the components as

$$\begin{aligned} u_0(t, x) &= e^{-x}, \\ u_1(t, x) &= \frac{e^{-x}t}{2^1 1!} (x - 2), \\ u_2(t, x) &= \frac{e^{-x}t^2}{2^2 2!} (x^2 - 6x + 6), \\ u_3(t, x) &= \frac{e^{-x}t^3}{2^3 3!} (x^3 - 12x^2 + 36x - 24), \\ &\vdots \\ u_j(t, x) &= \frac{(j + 1)!e^{-x}t^j}{2^j} \left[ \frac{x^j}{0!(j + 1)!j!} - \frac{x^{j-1}}{1!(j)!(j - 1)!} + \frac{x^{j-2}}{2!(j - 1)!(j - 2)!} \right. \\ &\quad \left. - \frac{x^{j-3}}{3!(j - 2)!(j - 3)!} \right. \\ &\quad \left. + \dots + (-1)^r \frac{x^{j-r}}{r!(j - r + 1)!(j - r)!} + \dots + (-1)^{j-1} \frac{x^1}{(j - 1)!(2)!(1)!} \right. \\ &\quad \left. + (-1)^j \frac{1}{(j)!(1)!(0)!} \right]. \end{aligned}$$

**Table 2** Error of example 4.6 when  $t = 1, x \in [0, 10]$

$n$	10	15	20	25	30	35	40
Error	9.56E-04	3.58E-05	1.30E-06	4.62E-08	1.57E-09	5.28E-11	1.86E-12



**Fig. 1** Comparison of  $\psi_n, n = 2, 4, 6, 8$  and the exact  $u$  with time  $t = 1$  of Example 4.6

Hence, the  $n$ -term truncated series solution can be obtained as

$$\psi_n(t, x) = \sum_{j=0}^n u_j(t, x).$$

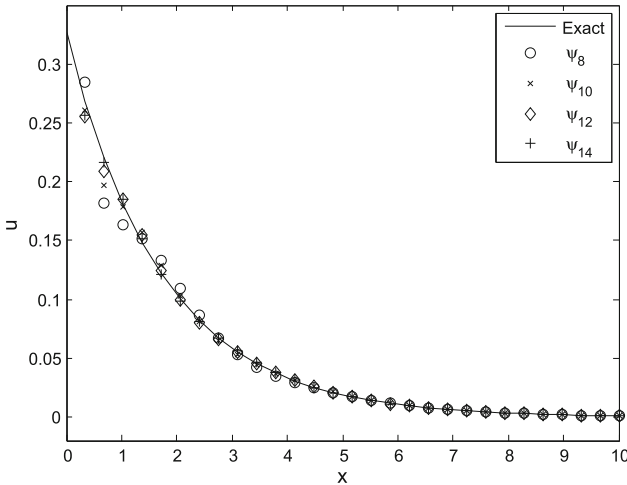
In order to check the accuracy of the ADM (3.6), we define error by fixing time  $t$  and letting  $x \in [0, 10]$ . We divide the interval  $[0, 10]$  into 100 subintervals  $[x_{i-1/2}, x_{i+1/2}]$ ,  $i = 1(1)100$ . Each subinterval has its representative  $x_i$  as the midpoint of the interval, i.e.,  $x_i = \frac{x_{i-1/2} + x_{i+1/2}}{2}$ . The grid points may be non-equispaced or equispaced such that  $x_{i+1/2} - x_{i-1/2} = h_i, i = 1(1)100$ . We define the error as

$$\text{Error} = \sum_{i=1}^{100} |\psi_n^i - u_i| h_i, \quad n = 10, 15, 20, \dots, 40. \tag{4.10}$$

We denote  $\psi_n^i = \psi_n(t, x_i)$  and  $u_i = u(t, x_i)$ .

Table 2 shows the error calculated by (4.10). As we can clearly see that as the number of terms in series solution increases the error decreases as expected by Theorem 3.2.

Figures 1 and 2 show the plots of the exact solution  $u(t, x)$  and the approximate series solution  $\psi_n(t, x)$  obtained by the ADM (3.6). Note that the curves of the approximate solutions  $\psi_8$  at time  $t = 1$  and  $\psi_{14}$  at  $t = 1.5$  are identical to the exact solution  $u(t, x)$ .



**Fig. 2** Comparison of  $\psi_n, n = 8, 10, 12, 14$  and the exact  $u$  with time  $t = 1.5$  of Example 4.6

**Table 3** Truncation error of Example 4.6

$m$	5	10	15	20	25	30
$\Delta_m$	0.0838802	0.0220186	0.00577988	0.00151722	0.000398271	0.000104546

Table 3 shows the truncation error of Example 4.6 using Theorem 3.3 as

$$\Delta_m := \frac{\delta^m}{(1 - \delta)} \|u_1\|, \quad \|u_1\| = \sup_{t \in [0, t_0]} \int_0^\infty t \frac{e^{-x}}{2} (x - 2) dx = t_0/2.$$

Here  $\delta := t_0 e^{2t_0 L} (\|u_0\| + 2t_0 L(L + 1))$ ;  $L = \|u_0\|(T + 1)$ ,  $\|u_0\| = \sup_{t \in [0, t_0]} \int_0^\infty e^{-x} dx = 1$ , and choosing  $T = 1$  and  $t_0 = 0.15$ .

*Example 4.7* Consider (1.3)–(1.4) with multiplicative kernel  $a(x, y) = xy$ ,  $u_0(x) = e^{-x}$ . Its analytic solution follows [39] as

$$u(t, x) = e^{-(t+1)x} \sum_{k=0}^\infty \frac{t^k x^{3k}}{(k + 1)! \Gamma(2k + 2)}.$$

According to the recursive scheme (3.6), we have

$$u_j(t, x) = \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x (x - y)y A_j dy - \int_0^\infty xy B_j dy \right], \quad j = 1, 2, \dots, \left. \vphantom{u_j(t, x)} \right\} \quad (4.11)$$

**Table 4** Error of Example 4.7 when  $t = 1, x \in [0, 10]$

$n$	10	15	20	25	30	35	40
Error	9.01E-01	1.29E-01	5.54E-02	2.41E-03	9.02E-06	5.32E-08	1.05E-10

Making use of (4.11) and (4.9), we obtain components as

$$\begin{aligned}
 u_0(t, x) &= e^{-x}, \\
 u_1(t, x) &= \frac{(tx)^1}{2!3!}(-12 + x^2)e^{-x}, \\
 u_2(t, x) &= \frac{(tx)^2}{3!5!} (360 - 60x^2 + x^4) e^{-x}, \\
 u_3(t, x) &= \frac{(tx)^3}{4!7!} (-20160 + 5040x^2 - 168x^4 + x^6) e^{-x}, \\
 &\vdots \\
 u_j(t, x) &= 2e^{-x}(xt)^j \left[ \frac{x^{2j}}{2(j+1)!(2j+1)!} - \frac{x^{2(j-1)}}{1!(j-1)!(2j)!} + \frac{x^{2(j-2)}}{2!(j-2)!(2j-1)!} \right. \\
 &\quad - \frac{x^{2(j-3)}}{3!(j-3)!(2j-2)!} + \dots + (-1)^r \frac{x^{2(j-r)}}{r!(j-r)!(2j-r+1)!} + \dots + \\
 &\quad \left. (-1)^{j-1} \frac{x^2}{(j-1)!(1)!(4)!} + (-1)^j \frac{1}{j!(0)!(2)!} \right].
 \end{aligned}$$

The  $n$ -term truncated series solution can be obtained as follows:

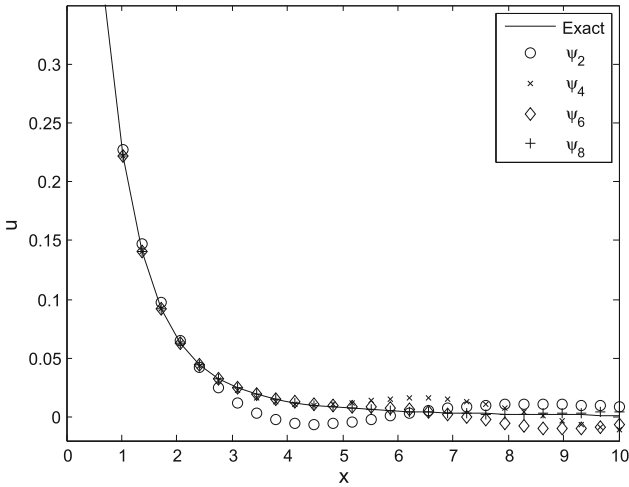
$$\psi_n(t, x) = \sum_{j=0}^n u_j(t, x).$$

Table 4 shows the error by using (4.10) for  $n = 10, 15, \dots, 40$  with  $t = 1$ . It is clear that as one increases the number of terms in the series solution the error decreases as expected.

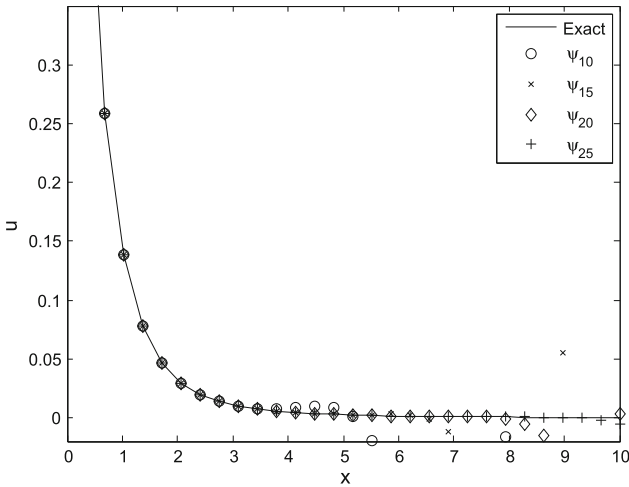
We plot the exact solution  $u(t, x)$  and the approximate series solution  $\psi_n(t, x)$  obtained by the ADM (3.6) in Figs. 3 and 4. It can be noted that the curve of the approximate solution  $\psi_8$  is almost identical to the exact solution  $u(t, x)$  with  $t = 0.5$ . Also, the approximate solution  $\psi_{25}$  and  $u(t, x)$  with  $t = 1$  are overlapping each others.

*Example 4.8* Consider (1.3)–(1.4) with multiplicative kernel  $a(x, y) = xy, u_0(x) = \frac{e^{-x}}{x}$ , which has exact solution as follows [9]

$$u(t, x) = \exp(-Tx) \frac{I_1(2x\sqrt{t})}{x^2\sqrt{t}},$$



**Fig. 3** Comparison of  $\psi_n, n = 2, 4, 6, 8$  and the exact  $u$  with time  $t = 0.5$  of Example 4.7



**Fig. 4** Comparison of  $\psi_n, n = 10, 15, 20, 25$  and the exact  $u$  with time  $t = 1$  of Example 4.7

where

$$T = \begin{cases} 1 + t, & t \leq 1, \\ 2\sqrt{t}, & \text{otherwise.} \end{cases}$$

Here  $I_1$  is the modified Bessel function of the first kind

$$I_1(x) = \frac{1}{\pi} \int_0^\pi \exp(x \cos \theta) \cos \theta d\theta. \tag{4.12}$$

**Table 5** Error of Example 4.8 when  $t = 1, x \in [0, 10]$

$n$	10	15	20	25	30	35	40
Error	1.79E-02	2.20E-03	4.01E-05	2.70E-07	1.15E-09	2.00E-12	4.18E-12

In view of the ADM (3.6), we have the recursive scheme as

$$u_j(t, x) = \mathcal{L}^{-1} \left[ \begin{aligned} &u_0(t, x) = \frac{e^{-x}}{x}, \\ &\left[ \frac{1}{2} \int_0^x (x-y)y A_j dy - \int_0^\infty xy B_j dy \right], \quad j = 1, 2, \dots, \end{aligned} \right]. \tag{4.13}$$

Using (4.13) and (4.9), we obtain components as

$$\begin{aligned} u_0(t, x) &= \frac{e^{-x} x^{-1}}{1}, \\ u_1(t, x) &= \frac{e^{-x} t}{2(1!)^2} (x - 2), \\ u_2(t, x) &= \frac{e^{-x} t^2 x}{3(2!)^2} (x^2 - 6x + 6), \\ u_3(t, x) &= \frac{e^{-x} t^3 x^2}{4(3!)^2} (x^3 - 12x^2 + 36x - 24), \\ &\vdots \\ u_j(t, x) &= e^{-x} x^{j-1} t^j \left[ \frac{x^j}{0!(j+1)!j!} - \frac{x^{j-1}}{1!(j)!(j-1)!} + \frac{x^{j-2}}{2!(j-1)!(j-2)!} \right. \\ &\quad - \frac{x^{j-3}}{3!(j-2)!(j-3)!} \\ &\quad \left. + \dots + (-1)^r \frac{x^{j-r}}{r!(j-r+1)!(j-r)!} + \dots + (-1)^{j-1} \frac{x^1}{(j-1)!(2)!(1)!} + (-1)^j \frac{1}{j!(1)!(0)!} \right]. \end{aligned}$$

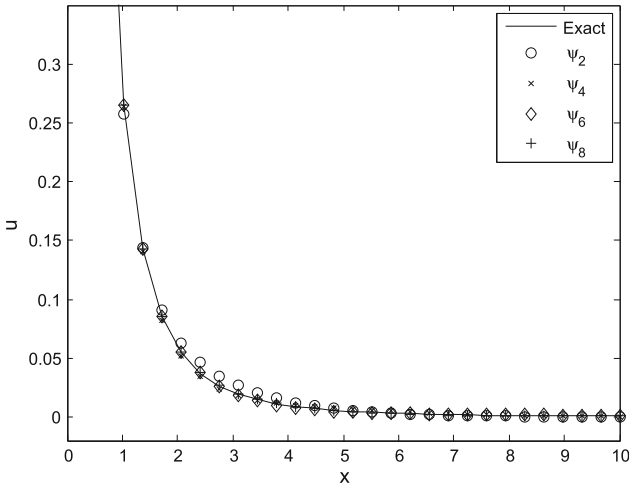
The  $n$ -term truncated series solution can be obtained as

$$\psi_n(t, x) = \sum_{j=0}^n u_j(t, x).$$

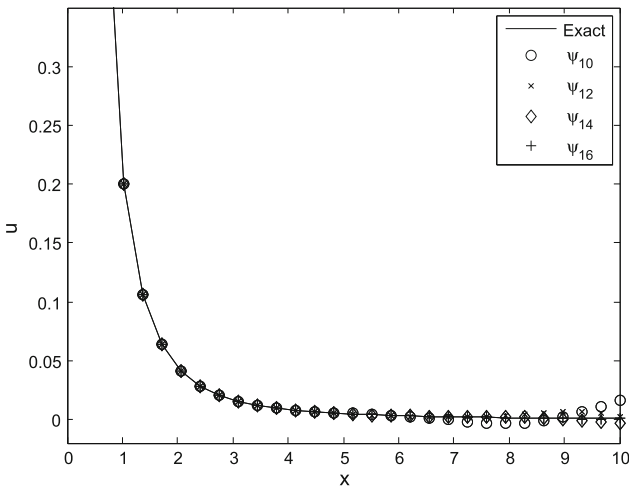
In Table 5, we have shown the error using the formula (4.10). As expected the error approaches to zero with the increase in number of terms in series solution.

In Figs. 5 and 6, we plot the exact solution  $u(t, x)$  and the approximate series solution  $\psi_n(t, x)$  obtained by the ADM (3.6) for  $t = 0.5, 1$ . It can be noted that the curves of the approximate solutions  $\psi_8$  at  $t = 0.5$  and  $\psi_{16}$  at  $t = 1$  are almost identical to the exact solution  $u(t, x)$ .





**Fig. 5** Comparison of  $\psi_n, n = 2, 4, 6, 8$  and the exact  $u$  with time  $t = 0.5$  of Example 4.8



**Fig. 6** Comparison of  $\psi_n, n = 10, 12, 14, 16$  and the exact  $u$  with  $t = 1$  of Example 4.8

*Example 4.9* Consider (1.3)–(1.4) with sum kernel  $a(x, y) = x + y, u_0(x) = e^{-x}$ , which has exact solution follows [39]

$$u(t, x) = \frac{(1 - \tau) \exp(-(1 + \tau)x)}{x\sqrt{\tau}} I_1(2x\sqrt{\tau}),$$

where  $\tau = 1 - e^{-t}$  and  $I_1(x)$  is the modified Bessel function of the first kind as in (4.12).

According to the ADM (3.6), we have

**Table 6** Error of Example 4.9 with time  $t = 1, x \in [0, 10]$

$n$	10	15	20	25	30	35	40
Error	9.77E-02	3.58E-03	3.05E-04	6.17E-05	5.66E-06	5.28E-08	1.86E-08

$$\left. \begin{aligned} u_0 &= e^{-x}, \\ u_j &= \mathcal{L}^{-1} \left[ \frac{1}{2} \int_0^x x A_j dy - \int_0^\infty (x+y) B_j dy \right], \quad j = 1, 2, \dots, \end{aligned} \right\} \quad (4.14)$$

Using the recursive scheme (4.14) and (4.9), we obtain solution components as follows:

$$\begin{aligned} u_0(t, x) &= e^{-x}, \\ u_1(t, x) &= \frac{1}{2} e^{-x} t (x^2 - 2x - 2), \\ u_2(t, x) &= \frac{1}{12} e^{-x} t^2 (6 + 18x - 3x^2 - 6x^3 + x^4), \\ u_3(t, x) &= \frac{1}{144} e^{-x} t^3 (-24 - 168x - 60x^2 + 120x^3 + 12x^4 - 12x^5 + x^6), \\ &\vdots \end{aligned}$$

In a similar fashion, we list the error of the Example 4.9 using (4.10) in Table 6. It is obvious from these that the error decreases as one increases the terms in series solution.

### 5 Concluding remarks

In this paper we have shown the application of Adomian decomposition method for solving the fragmentation and the aggregation population balance equations. The ADM provides a direct recursive scheme for obtaining the exact as well as approximate solutions. It is well known that the ADM allows to solve both nonlinear initial and boundary value problems without unphysical restrictive assumptions such as linearization, discretization, perturbation and guessing the initial term or a set of basis functions. Convergence of the series solution has been discussed. Convergence analysis is reliable enough to estimate the maximum absolute truncated error of the series solution. We have examined the accuracy and the performance of the method by solving several examples. It is also worth noting that in this work, we have been able to obtain some exact solutions of the fragmentation population balance equations for some kernels and selection functions. It has been observed that only few terms of the series solution are enough to obtain accurate approximations to the solution.

## References

1. Randolph, A.D.: A population balance for countable entities. *Can. J. Chem. Eng.* **42**(6), 280–281 (1964)
2. Hulburt, H., Katz, S.: Some problems in particle technology: a statistical mechanical formulation. *Chem. Eng. Sci.* **19**(8), 555–574 (1964)
3. Dorao, C., Jakobsen, H.: Time-property least-squares spectral method for population balance equations. *J. Math. Chem.* **46**(3), 770–780 (2009)
4. Ziff, R.: New solutions to the fragmentation equation. *J. Phys. A: Math. General* **24**, 2821 (1991)
5. Kumar, R., Kumar, J.: Numerical simulation and convergence analysis of a finite volume scheme for solving general breakage population balance equations. *Appl. Math. Comput.* **219**(10), 5140–5151 (2013)
6. Kumar, J., Peglow, M., Warnecke, G., Heinrich, S., Mörl, L.: Improved accuracy and convergence of discretized population balance for aggregation: the cell average technique. *Chem. Eng. Sci.* **61**(10), 3327–3342 (2006)
7. Filbet, F., Laurençot, P.: Numerical simulation of the smoluchowski coagulation equation. *SIAM J. Sci. Comput.* **25**(6), 2004–2028 (2004)
8. Hammouch, Z., Mekkaoui, T.: A laplace-variational iteration method for solving the homogeneous smoluchowski coagulation equation. *Appl. Math. Sci.* **6**(18), 879–886 (2012)
9. Ranjbar, M., Adibi, H., Lakestani, M.: Numerical solution of homogeneous smoluchowski's coagulation equation. *Int. J. Comput. Math.* **87**(9), 2113–2122 (2010)
10. Yıldırım, A., Koçak, H.: Series solution of the smoluchowski's coagulation equation. *J. King Saud Univ.-Sci.* **23**(2), 183–189 (2011)
11. Singh, M., Chakraborty, J., Kumar, J., Ramakanth, R.: Accurate and efficient solution of bivariate population balance equations using unstructured grids. *Chem. Eng. Sci.* **93**, 1–10 (2013)
12. Abulwafa, E., Abdou, M., Mahmoud, A.: The solution of nonlinear coagulation problem with mass loss. *Chaos Solitons Fract.* **29**(2), 313–330 (2006)
13. Kumar, J., Peglow, M., Warnecke, G., Heinrich, S.: An efficient numerical technique for solving population balance equation involving aggregation, breakage, growth and nucleation. *Powder Technol.* **182**(1), 81–104 (2008)
14. Ramkrishna, D.: The status of population balances. *Rev. Chem. Eng.* **3**(1), 49–95 (2011)
15. McLaughlin, D., Lamb, W., McBride, A.: Existence results for non-autonomous multiple-fragmentation models. *Math. Methods Appl. Sci.* **20**(15), 1313–1323 (1997)
16. Dubovskii, P., Galkin, V., Stewart, I.: Exact solutions for the coagulation-fragmentation equation. *J. Phys. A Math. General* **25**, 4737 (1992)
17. Ziff, R., McGrady, E.: The kinetics of cluster fragmentation and depolymerisation. *J. Phys. A Math. General* **18**, 3027 (1985)
18. Hasseine, A., Bellagoun, A., Bart, H.-J.: Analytical solution of the droplet breakup equation by the adomian decomposition method. *Appl. Math. Comput.* **218**(5), 2249–2258 (2011)
19. Attarakih, M.M., Bart, H.-J., Faqir, N.M.: Solution of the droplet breakage equation for interacting liquid-liquid dispersions: a conservative discretization approach. *Chem. Eng. Sci.* **59**(12), 2547–2565 (2004)
20. Biazar, J., Ayati, Z., Yaghouti, M.R.: Homotopy perturbation method for homogeneous smoluchowski's equation. *Numer. Methods Partial Differ. Equ.* **26**(5), 1146–1153 (2010)
21. Efati, S.: Solution of the smoluchowski's equation by homotopy analysis method. *Int. J. Nonlinear Sci.* **11** (2011)
22. Al-Khaled, K., Allan, F.: Decomposition method for solving nonlinear integro-differential equations. *J. Appl. Math. Comput.* **19**(1), 415–425 (2005)
23. El-Kalla, I.: Error estimates for series solutions to a class of nonlinear integral equations of mixed type. *J. Appl. Math. Comput.* **38**(1), 341–351 (2012)
24. Momani, S., Moadi, K.: A reliable algorithm for solving fourth-order boundary value problems. *J. Appl. Math. Comput.* **22**(3), 185–197 (2006)
25. Singh, R., Kumar, J., Nelakanti, G.: Numerical solution of singular boundary value problems using Green's function and improved decomposition method. *J. Appl. Math. Comput.* **43**(1–2), 409–425 (2013)
26. Singh, R., Kumar, J.: The Adomian decomposition method with greens function for solving nonlinear singular boundary value problems. *J. Appl. Math. Comput.* **44**(1–2), 397–416 (2014)

27. Rach, R., Duan, J.-S., Wazwa, A.-M.: Solving coupled Lane-Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method. *J. Math. Chem.* **52**(1), 255–267 (2013)
28. Duan, J.-S.: Convenient analytic recurrence algorithms for the adomian polynomials. *Appl. Math. Comput.* **217**(13), 6337–6348 (2011)
29. Duan, J.: New recurrence algorithms for the nonclassic adomian polynomials. *Comput. Math. Appl.* **62**(8), 2961–2977 (2011)
30. Duan, J.: An efficient algorithm for the multivariable adomian polynomials. *Appl. Math. Comput.* **217**(6), 2456–2467 (2010)
31. Duan, J.: Recurrence triangle for adomian polynomials. *Appl. Math. Comput.* **216**(4), 1235–1241 (2010)
32. Rach, R.C.: A new definition of the adomian polynomials. *Kybernetes* **37**(7), 910–955 (2008)
33. Wazwaz, A.: A new method for solving singular initial value problems in the second-order ordinary differential equations. *Appl. Math. Comput.* **128**(1), 45–57 (2002)
34. Cherruault, Y., et al.: A comparison of numerical solutions of fourth-order boundary value problems. *Kybernetes* **34**(7/8), 960–968 (2005)
35. Adomian, G., Rach, R.: Inversion of nonlinear stochastic operators. *J. Math. Anal. Appl.* **91**(1), 39–46 (1983)
36. Adomian, G.: *Solving Frontier Problems of Physics: The Decomposition Method [ie Method]*. Kluwer Academic Publishers, Berlin (1994)
37. Khuri, S., Sayfy, A.: A novel approach for the solution of a class of singular boundary value problems arising in physiology. *Math. Comput. Modell.* **52**(3), 626–636 (2010)
38. Stewart, I., Meister, E.: A global existence theorem for the general coagulation-fragmentation equation with unbounded kernels. *Math. Methods Appl. Sci.* **11**(5), 627–648 (1989)
39. Kumar, J.: Numerical approximations of population balance equations in particulate systems, Ph.D. thesis, Otto-von-Guericke-Universität Magdeburg, Universitätsbibliothek (2006)
40. Austin, L.G.: A treatment of impact breakage of particles. *Powder Technol.* **126**(1), 85–90 (2002)