SYNOPSIS OF

A STUDY ON CONDITIONAL (k,r)-COLORING
AND RADIO LABELING OF GRAPHS

A THESIS

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INTRODUCTION

The study on graph colorings and labelings plays an essential role in graph theory. Time tabling, sequencing, and scheduling problems, in their many forms, are basically of graph coloring nature. Frequency assignment problem can be modeled as a distance constrained labeling of graphs. In the thesis we study a variant of graph coloring called conditional \((k, r)\)-coloring and a type of labeling called radio labeling of graphs. First we present some basic definitions and notations that will be used throughout the thesis.

Let \(G = (V, E)\) be a simple, connected, undirected graph. For a vertex \(v \in V\), the \((open)\) neighborhood of \(v\) in \(G\) is defined as \(N(v) = \{u \in V : (u, v) \in E\}\) and the \((closed)\) neighborhood of \(v\) is defined as \(N[v] = N(v) \cup \{v\}\). The subgraph induced by a set \(A \subseteq V\) is denoted by \(G[A]\). The Cartesian product \(G \square H\) of graphs \(G\) and \(H\) has the vertex set \(V(G) \times V(H)\) and the edge set \(E(G \square H) = \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in E(G)\) and \(x_2 = y_2, \) or \((x_2, y_2) \in E(H)\) and \(x_1 = y_1\}\). The strong product \(G \otimes H\) of graphs \(G\) and \(H\) is defined as follows: \(V(G \otimes H) = V(G) \times V(H)\) and \(E(G \otimes H) = E_c \cup E_d\), where \(E_c = \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in E(G)\) and \(x_2 = y_2, \) or \((x_2, y_2) \in E(H)\) and \(x_1 = y_1\}\) and \(E_d = \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in E(G)\) and \((x_2, y_2) \in E(H)\}\). The join \(G + H\) has the vertex set \(V(G) \cup V(H)\) and the edge set \(E(G) \cup E(H) \cup CE\) where the cross-edge set \(CE = \{(u_g, u_h) : u_g \in V(G)\) and \(u_h \in V(H)\}\).

A \(k\)-partite graph is a one whose vertex set \(V\) can be partitioned into \(k\) disjoint non-empty subsets such that no two vertices in the same subset are adjacent. A complete \(p\)-partite graph is a \(k\)-partite graph such that every pair of vertices from two different subsets are adjacent. By \(K[n_1, \ldots, n_p]\) we denote a complete \(p\)-partite graph where, for \(1 \leq i \leq p\), the \(i^{th}\) partition has \(n_i\) vertices. We take \(n = \sum_{j=1}^{p} n_j\). The Windmill graph denoted by \(Wd(k, n)\) consists of \(n\) copies of \(K_k\) and identifying one vertex from each \(K_k\) copy as a common center vertex. We denote \(Wd(3, n)\) by \(F_n\), the friendship graph. The middle graph \(M(G)\) of \(G\) is the graph whose vertex set corresponds to \(V(G) \cup\)
in $M(G)$ two vertices are adjacent iff one of the following is satisfied:

(a) they are adjacent edges of $G$.
(b) one is a vertex and the other is an edge incident with it.

The total graph $T(G)$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent iff (a) or (b) above holds or they are adjacent vertices of $G$.

The results obtained in the thesis are briefly explained here.

i. Exact values and bounds for $\chi_r(G)$ when $G$ is a graph obtained by some graph operations are obtained. Also a simple $O(nr^2)$ algorithm for $\chi_r(G)$ when $G$ is $r$-regular of a fixed order $n$ is presented.

ii. For specific values of $r$, $r^{th}$-order conditional chromatic number of a graph $G$ when $G$ is a line, middle and total graph of some parameterized graph is obtained.

iii. Generalized the notion of unique colorability of graphs and defined unique conditional colorability; showed that some graphs are uniquely ($k, r$)-colorable for some integers $k, r$. The notion of dynamic star coloring of graphs is introduced and a few results on dynamic star chromatic number of graphs is obtained.

iv. A distance-2-clique-set or D2CS of a graph $G$ is defined and established its relation to $\chi_r(G)$. We have showed that if the number of D2CSs in a graph is $k$ then all these D2CSs can be enumerated in time $O(kn^3/\log^2 n)$. A recursive algorithm for finding the size of a maximum D2CS in a graph is presented. We have also showed that all maximal D2CSs in a strongly chordal graph on $n$ vertices can be enumerated in time $O(n)$.

v. An upper bound for the radio number of binomial tree, Fibonacci trees, uniform caterpillar and strong product of paths is derived.

**CONDITIONAL ($k, r$)-COLORING OF GRAPHS**

For integers $k, r > 0$, a conditional ($k, r$)-coloring of a graph $G$ is a surjective mapping $c: V \to \{1, \ldots, k\}$ satisfying the following conditions:

(C1) if $(u, v) \in E$, then $c(u) \neq c(v)$;
(C2) for any $v \in V$, $|c(N(v))| \geq \min \{r, d(v)\}$, where $c(U) = \{c(u) : u \in U \text{ for a set } U \subseteq V\}$.

For a given integer $r > 0$, the smallest integer $k$ such that $G$ has a conditional ($k, r$)-coloring is called the $r^{th}$-order conditional chromatic number of $G$, denoted by $\chi_r(G)$.
In particular when \( r = 2 \) this coloring is called \textit{dynamic coloring} and the corresponding chromatic number is called \textit{dynamic chromatic number} denoted by \( \chi_d(G) \) or \( \chi_2(G) \). Clearly \( \chi_1(G) = \chi(G) \), hence conditional \((k, r)\)-coloring is a generalization of traditional coloring in some way. Given a graph \( G \) and an integer \( k > 0 \), the complexity of deciding whether \( \chi(G) \leq k \) is NP-complete [12].

We define the problem \( \text{COND-}k-r\text{-COL} \) as: Given a graph \( G \) and integers \( k, r > 0 \) is it possible to obtain a conditional \((k, r)\)-coloring of \( G \)?

**Theorem 1.** [16] \( \text{COND-}k-r\text{-COL} \) is NP-complete.

In [16] it is proved that conditional \((3, 2)\)-coloring is NP-complete even for planar bipartite graphs with \( \Delta(G) \leq 3 \) and this is not influenced by the girth of \( G \). Few results on dynamic coloring of graphs are available in the literature (see for example [13, 1, 2, 3]). Very few results are available on conditional \((k, r)\)-coloring of graphs [14, 16, 15]. We have obtained exact values for \( \chi_r(G) \) for some values of \( r \) when \( G \) is restricted to a specific class of graphs. We also give bounds on \( \chi_r(G) \) when \( G \) is obtained by some graph operations. We begin with some basic results on conditional \((k, r)\)-coloring.

First we show that the value of \( \chi_r(G) \) (i) can depend only on some parameter of \( G \) or (ii) can depend only on \( r \) or (iii) is equal to \( n = |V| \). The following theorem asserts this by taking \( G \cong K[n_1, \ldots, n_p] \).

**Theorem 2.** Let \( G = K[n_1, \ldots, n_p] \) be a complete \( p \)-partite graph where, for \( 1 \leq i \leq p \), the \( i \)th partition has \( n_i \) vertices. We take \( n = \sum_{j=1}^{p} n_j \). Then the following hold.

(i) If \( n_i \geq 2 \) for at least two \( i \)’s then \( \chi_p(G) = p + 2 \).
(ii) If \( p < r \leq 2p - 2 \) and for each \( 1 \leq i \leq p \), \( n_i \geq 2 \) then \( \chi_r(G) = r + 2 \).
(iii) If \( n - \min\{n_1, \ldots, n_p\} \leq r \) then \( \chi_r(G) = n \).

Next we give some class of graphs for which \( \chi_{\Delta}(G) = \Delta + 1 \neq \chi(G) \). Let the property \( \text{KN-LIKE}(G, \Delta) \) be defined as:

\[
\chi_{\Delta}(G) = \Delta + 1
\]

Now we ask the question, if \( \text{KN-LIKE}(G, \Delta) \) holds in \( G \) does it imply that \( G \) is complete? That the answer is negative is asserted by the following theorem.

**Theorem 3.** \( \text{KN-LIKE}(G, \Delta) \) holds for the following graphs.

(i) \( G(2, n) = P_2 
\square P_n \)
(ii) \( C_n^2 \) when \( n > 9 \) and \( n \equiv 0 \pmod{5} \)
(iii) For \( n, m \geq 2 \), \( P_n \otimes P_m \)
(iv) Line and middle graphs of complete \( k \)-ary tree \( T \) with height \( h \geq 2 \).
(v) Total graph of $W_n$.

The following three theorems give either the value of $\chi_r(G)$ or upper bound when $G$ is obtained by some graph operation for specific values of $r$.

**Theorem 4.** Given any two graphs $G_1$ and $G_2$, let $r_1$ and $r_2$ be such that $r_1 \geq \delta(G_1)$ and $r_2 \geq \delta(G_2)$. Then $\chi_r(G_1 \Box G_2) \leq \chi_{r_1}(G_1) \ast \chi_{r_2}(G_2)$ where $r \leq \delta(G_1) + \delta(G_2)$.

**Theorem 5.** Let $G_1$ and $G_2$ be two graphs where $\chi(G_1) = k_1$, $\chi(G_2) = k_2$ and w.l.o.g. let $k_1 \leq k_2$. Then $\chi_r(G_1 + G_2) = \chi(G_1 + G_2) = k_1 + k_2$, where $r \leq k_1 + 1$.

**Theorem 6.** Let $T_1, T_2$ be two non trivial trees with $n_1, n_2$ number of vertices respectively and w.l.o.g. let $n_1 \leq n_2$. Then $\chi_r(T_1 + T_2) = 2(r - 1)$, where $4 \leq r \leq n_1 + 1$.

The following theorem gives value of $\chi_r(G)$ for some $r$ when $G$ is bipartite.

**Theorem 7.** Let $G(V_1, V_2, E)$ be a bipartite graph, $S_1 = \bigcap_{u \in V_1} N_G(u)$, $S_2 = \bigcap_{v \in V_2} N_G(v)$ and w.l.o.g. let $|S_1| \leq |S_2|$. Then $\chi_r(G) = 2r$ where $r \leq |S_1|$.

From [24] we know that there exists a backtracking graph coloring algorithm whose average-case time complexity is in $O(1)$. Based on this algorithm we have given a simple algorithm for obtaining $\chi_r(G)$ when $G$ is $r$-regular and hence have the following result.

**Theorem 8.** The average-case time-complexity of obtaining $\chi_r(G)$ when $G$ is $r$-regular of a fixed order $n$ is in $O(nr^2)$.

We define a type of web graph as follows.

**Definition 1.** For $t \geq 1$ and $n \geq 3$, by $W(t, n)$ we denote the graph consisting of $t$ induced cycles $C_n$ such that no two $C_n$’s have a vertex in common and is constructed recursively as follows. Let $W(1, n) = W_{n+1}$, the wheel graph on $n + 1$ vertices. We denote by $v_{0,0}$ the vertex of $W(1, n)$ with degree $n$. Assume that $W(t, n)$ has been obtained. Let $v_{t,1}, v_{t,2}, \ldots, v_{t,n}$ be the induced cycle in $W(t, n)$ such that for all $1 \leq i \leq n$, $d(v_{t,i}) = 3$ and $v_{t,i}$ is adjacent to $v_{t,j}$ only if $|i - j| = 1$ or $n - 1$. We construct $W(t + 1, n)$ from $W(t, n)$ with the following vertex and edge sets:

$$V(W(t + 1, n)) = V(W(t, n)) \cup \{v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,n}\} \text{ and}$$

$$E(W(t + 1, n)) = E(W(t, n)) \cup \{(v_{t,i}, v_{t+1,i}) : 1 \leq i \leq n\} \cup E',$$

where $E' = \{(v_{t+1,i}, v_{t+1,j}) : 1 \leq i, j \leq n \text{ and } j - i = 1 \text{ or } n - 1\}$.

The following theorem gives the dynamic chromatic number of web graph, gear graph and middle graph of cycle.

**Theorem 9.** The following holds.

(i) For $t \geq 1$ and $n \geq 3$, $\chi_d(W(t, n)) = 4$.

(ii) For $n \geq 3$ let $G_n$ be the $n$-gear then $\chi_d(G_n) = 4$.

(iii) For $n \geq 4$, let $M(C_n)$ be the middle graph of $C_n$. Then $\chi_d(M(C_n)) = 3$. 


CONDITIONAL CHROMATIC NUMBER OF SOME PARAMETERIZED GRAPHS

In the following theorems we give $r$th-order conditional chromatic number of some parameterized graphs. We begin with the square of cycle.

**Theorem 10.** Let $C_n^2$ be the square of $C_n$ such that $n \geq 3$ and $n \neq 13, 14$ and $19$. Then

$$
\chi_r(C_n^2) = \begin{cases} 
n, & \text{if } n \leq 5 \text{ or both } 6 \leq n \leq 9 \text{ and } r = 4. \\
4, & \text{if } n \not\equiv 3 \pmod{4} \text{ and } r = 3. \\
6, & \text{if } n > 9, n \not\equiv 0 \pmod{5} \text{ and } r = 4.
\end{cases}
$$

The following theorem gives 3rd-order conditional chromatic number of middle graph of cycle.

**Theorem 11.** For $n \geq 4$, let $M(C_n)$ be the middle graph of $C_n$. Then $\chi_r(M(C_n)) = 4$ when $r = 3$.

The following two theorems give $r$th-order conditional chromatic number of middle graph of complete bipartite graph and complete $p$-partite graphs for different values of $r$.

**Theorem 12.** Let $M(K_{n_1, n_2})$ be the middle graph of complete bipartite graph $K_{n_1, n_2}$ and w.l.o.g. assume $n_1 \leq n_2$. Then

$$
\chi_r(M(K_{n_1, n_2})) = \begin{cases} 
n_2 + 1, & \text{if } r \leq n_2. \\
n_2 + 2, & \text{if } r = n_2 + 1.
\end{cases}
$$

**Theorem 13.** Let $M(K[n_1, \ldots, n_p])$ be the middle graph of $K[n_1, \ldots, n_p]$. Then

$$
\chi_{\Delta}(M(K[n_1, \ldots, n_p])) = p + l.
$$

where $n = \sum_{i=1}^{p} n_i$ and $l = 1/2 \sum_{i=1}^{p} n_i(n - n_i)$.

The following theorem gives $\chi_r(G)$ when $G$ is gear graph.

**Theorem 14.** For $n \geq 3$, let $G_n$ be the $n$-gear graph. Then

$$
\chi_r(G_n) = \begin{cases} 
\chi_2(C_{2n}) + 1, & \text{if } r = 3. \\
\min\{r, \Delta\} + 1, & \text{if } r \geq 4.
\end{cases}
$$

In [14] for any integer $n$ the value $\chi_2(C_n)$ is obtained.

The following three theorems give $\chi_r(G)$ when $G$ is windmill, line graph of windmill and total graph of windmill graph.
Theorem 15. For integers $k \geq 3$ and $n \geq 2$, let $W_d(k,n)$ be a windmill graph. Then
\[
\chi_r(W_d(k,n)) = \begin{cases} 
  k, & \text{if } 2 \leq r \leq k - 1. \\
  \min\{r, \Delta\} + 1, & \text{if } r \geq k.
\end{cases}
\]

Theorem 16. For integers $k \geq 3$ and $n \geq 2$, let $L(W_d(k,n))$ be the line graph of windmill graph $W_d(k,n)$. Then
\[
\chi_{\Delta}(L(W_d(k,n))) = n(k - 1) + \left(\frac{k - 1}{2}\right) = z \text{ (say)}.
\]

Theorem 17. For integers $k \geq 3$ and $n \geq 2$, let $T(W_d(k,n))$ be the total graph of windmill graph $W_d(k,n)$ and $r \geq (n + 1)(k - 1)$. Then
\[
\chi_r(T(W_d(k,n))) = \begin{cases} 
  \min\{r, \Delta\} + 1, & \text{if } r \geq n(k - 1) + \left(\frac{k}{2}\right) + 1 \text{ and } k < 2n. \\
  n(k - 1) + \left(\frac{k}{2}\right) + 1, & \text{otherwise}.
\end{cases}
\]

As a corollary of above theorem we get the following result.

Corollary 18. Let $T(F_n)$ be the total graph of friendship graph $F_n$. Then
\[
\chi_r(T(F_n)) = \begin{cases} 
  2n + 4, & \text{if } r = 2n + 2 \text{ or } 2n + 3. \\
  \min\{r, \Delta\} + 1, & \text{if } r \geq 2(n + 2).
\end{cases}
\]

The theorem given below gives $\chi_r(G)$ when $G$ is the middle graph of friendship graph.

Theorem 19. Let $M(F_n)$ be the middle graph of friendship graph $F_n$. Then
\[
\chi_r(M(F_n)) = \begin{cases} 
  2n + 1, & \text{if } r \leq 2n. \\
  2n + 2, & \text{if } r = 2n + 1. \\
  2n + 4, & \text{if } r = \Delta.
\end{cases}
\]

When $G$ is total graph of wheel graph and middle graph of wheel graph the following two theorems give $\chi_r(G)$ for some $r$.

Theorem 20. For $n \geq 4$, let $T(W_n)$ be the total graph of $W_n$. Then
\[
\chi_r(T(W_n)) = \begin{cases} 
  3n - 2, & \text{if } r = \Delta \text{ and } n < 6. \\
  2n - 1, & \text{if } r = \Delta \text{ and } n \geq 6. \\
  n + 5, & \text{if } r = n + 2 \text{ and } n \equiv 1 \pmod{5}. \\
  n + 6, & \text{if } r = n + 2, n \not\equiv 1 \pmod{5} \text{ and } n \neq 8. \\
  n, & \text{if } r \leq n - 1 \text{ and } n > 7.
\end{cases}
\]

Theorem 21. Let $M(W_n)$ be the middle graph of $W_n$. Then
\[
\chi_r(M(W_n)) = \begin{cases} 
  10, & \text{if } r = \Delta \text{ and } n = 4. \\
  11, & \text{if } r = \Delta \text{ and } n = 5. \\
  n + 5, & \text{if } r = \Delta \text{ and } n \not\equiv 3 \pmod{5}.
\end{cases}
\]
UNIQUE CONDITIONAL COLORABILITY AND DYNAMIC STAR COLORING OF GRAPHS

We extend the notion of unique colorability to conditional \((k, r)\)-coloring. If \(\chi(G) = k\) and every \(k\)-coloring of \(G\) induces the same partition of \(V(G)\) then \(G\) is called uniquely \(k\)-colorable. In a similar way we define unique \((k, r)\)-colorability of graphs.

**Definition 2.** If \(\chi_r(G) = k\) and every conditional \((k, r)\)-coloring of \(G\) induces the same partition of \(V(G)\), then \(G\) will be called uniquely \((k, r)\)-colorable.

With the following propositions we explore further unique \((k, r)\)-coloring.

**Proposition 22.** If \(G\) is uniquely \(p\)-colorable and \(r \leq p - 1\) then \(\chi_r(G) = p\).

The definition of conditional \((k, r)\)-coloring of graph and the above proposition together imply:

**Corollary 23.** Every uniquely \(p\)-colorable graph \(G\) is also uniquely \((p, p-1)\)-colorable.

The following result shows that it is possible to construct uniquely \((3, 2)\)-colorable graph on any number of vertices.

**Proposition 24.** For every \(k \geq 3\), there exists a uniquely \((3, 2)\)-colorable graph \(G_k\) with \(k + 2\) vertices.

Unique colorability of path is shown in the following result.

**Proposition 25.** Every path \(P_n\) \((n \geq 3)\) is uniquely \((3, 2)\)-colorable.

The following result discusses the unique conditional colorability of trees.

**Proposition 26.** Let \(T \neq P_n\) be a rooted tree with \(n\) vertices and \(k = \chi_r(T)\) then \(T\) is not uniquely \((k, r)\)-colorable unless \(k = n\).

By combining the notions of dynamic coloring and star coloring of graphs we introduce the notion of dynamic star coloring of graphs. A dynamic star coloring of an undirected graph \(G\) is a proper vertex coloring of \(G\) such that any path of length three in \(G\) is not bicolored, and each vertex of degree greater than one has at least two distinctly colored neighbors. More formally, for an integer \(k > 0\), a dynamic star coloring of a graph \(G\) is a surjective mapping \(c: V(G) \to \{1, \ldots, k\}\) satisfying the following three properties:

1. **(P1)** if \((u, v) \in E(G)\), then \(c(u) \neq c(v)\);
2. **(P2)** for any vertex \(v \in V(G)\), \(|c(N(v))| \geq \min\{2, d(v)\}\), where \(c(S) = \{c(u) : u \in S\} \subseteq V(G)\).
3. **(P3)** for any \(P_4 \subseteq G\), \(|c(V(P_4))| > 2\).
The smallest integer $k$ such that $G$ has a dynamic star coloring is called the dynamic star chromatic number of $G$, denoted by $\chi_{ds}(G)$. Clearly the properties (P1) and (P2) define dynamic coloring. Property (P3) defines the star coloring property of graphs. The smallest integer $k$ such that $G$ has a star coloring is called the star chromatic number of $G$, denoted by $\chi_s(G)$. With the following results we explore further dynamic star coloring.

**Theorem 27.** For any tree $T$, $\chi_{ds}(T) = 3$.

**Theorem 28.** For $n \geq 3$, let $W_n$ be a wheel graph on $n$ vertices. Then

$$\chi_{ds}(W_n) = \begin{cases} 5, & \text{if } n = 6. \\ 4, & \text{otherwise.} \end{cases}$$

**Theorem 29.** For integers $n, m > 1$ let $K_{m,n}$ be a complete bipartite graph. Then $\chi_{ds}(K_{m,n}) = \min \{m, n\} + 2$.

**Theorem 30.** For $n \geq 2$, let $P_2 + P_n$ be the join of graphs $P_2$ and $P_n$. Then

$$\chi_{ds}(P_2 + P_n) = \begin{cases} 4, & \text{if } n = 2 \text{ or } 3. \\ 5, & \text{otherwise.} \end{cases}$$

**Theorem 31.** Let $G \Box H$ be the Cartesian product of graphs $G$ and $H$. Then $\chi_{ds}(G \Box H) \leq \chi_{ds}(G) * \chi_{ds}(H)$.

**Theorem 32.** For $n \geq 3$, let $K_2 \Box C_n$ be the Cartesian product of graphs $K_2$ and $C_n$. Then $\chi_{ds}(K_2 \Box C_n) = 4$.

**D2CS**

We define a distance-2-clique-set or D2CS of a graph $G$ to be a subset $S$ of $V$ such that every two distinct vertices in $S$ are at a distance at most 2 in $G[S]$ i.e., $\text{diam}(G[S]) \leq 2$. This notion is found to be useful in our arguments concerning bounds on $\chi_r(G)$ for some specific cases of $G$. The number of D2CSs in any other graph with $n$ vertices lies between $n + 1$ and $2^n$. For example the complete graph $K_n$, the ladder graph $L_n \cong P_n \Box P_2$ and the complement of complete graph $\overline{K_n}$ have $2^n$, $10n - 6$ and $n + 1$ D2CSs respectively. A maximum D2CS is one which has the largest size among all D2CSs. By $\omega_2(G)$ we denote the size of a maximum D2CS. Clearly $\Delta + 1 \leq \omega_2(G) \leq \Delta^2 + 1$ and these bounds are tight. For example $\omega_2(K_{1,n}) = \Delta(K_{1,n}) + 1 = n + 1$ and $\omega_2(C_5) = \Delta^2(C_5) + 1 = 5$. Unlike a clique every subset of a D2CS need not be a D2CS. Also a maximum D2CS need not contain as induced subgraph a maximum clique. As
a practical example consider $G$ to be a social network where $V$ represents people and $E$ captures mutual acquaintances. To find a largest group of people who all know each other or have a common acquaintance, one needs to find a maximum size D2CS.

The following definition categorizes D2CSs into two types.

**Definition 3:** Given a D2CS $S$ of $G$, w.r.t. a vertex $v \in S$, we define property $\text{ONE-DIST}(v)$ as:

$$\text{for all } u(\neq v) \in S, d(v, u) = 1.$$ 

*Type-1 D2CS:* A D2CS for which $\text{ONE-DIST}(v)$ is true for some vertex $v$.

*Type-2 D2CS:* A D2CS for which $\text{ONE-DIST}(v)$ is false for every vertex $v$.

We recall the following definitions.

**Definition 4.** A Fibonacci tree, a variant of binary tree is defined recursively as follows:

(a) Fibonacci tree of order $0$ and $1$ is a single node.

(b) Fibonacci tree of order $n$ ($n \geq 2$) is constructed by attaching a Fibonacci tree of order $n - 2$ as the leftmost child of the Fibonacci tree of order $n - 1$.

**Definition 5.** A binary Fibonacci tree of order $n$ ($n > 1$) is a variant of a binary tree defined recursively as follows:

(a) Binary Fibonacci tree of order $0$ has a single node and order $1$ is $P_2$.

(b) Binary Fibonacci tree of order $n$ is a variant of binary tree whose left subtree is a binary Fibonacci tree of order $n - 1$ and whose right subtree is a binary Fibonacci tree of order $n - 2$.

**Definition 6.** A binomial tree $B_n$ of order $n$ ($n \geq 0$) is an ordered tree defined recursively as follows:

(a) $B_0$ is a one-vertex graph.

(b) $B_n$ consists of two copies of $B_{n-1}$ such that the root of one is the leftmost child of the root of the other.

We first give some results on enumerating D2CSs in some specific graphs based on combinatorial aspects which make the concept of D2CS more clear.

**Proposition 33.** Let $f(n)$, $g(n)$, $h(n)$ be the number of D2CSs in a Fibonacci tree, binary Fibonacci tree and binomial tree of order $n$ respectively. Then

(i) For $n \geq 4$, $f(n) = 3 \times 2^{n-2} - F_{n-1} - F_{n+1} + 2$.

(ii) For $n \geq 4$, $g(n) = 2F_n + 3F_{n+2} - 9$.

(iii) For $n \geq 1$, $h(n) = n2^n + 2$.

**Definition 7.** A graph is a split if its vertex set can be partitioned into a clique and an independent set.
Proposition 34. Let $G$ be a split graph with $K \subseteq V(G)$, $|K| = \omega(G) = k$, for all $v \in K, d(v) = k + r - 1$ and for all $v' \in V(G) \setminus K, d(v') = 1$. Then the number of $D2CSs$ of $G$ is $k^2(k+1)(2r-1) + 2k + kr$.

Algorithm for Counting and Enumerating the $D2CSs$ of a Graph

We give an algorithm for enumerating $D2CSs$ based on the fact that, if $G[S]$ is a $D2CS$ in $G$ then $H[S]$ is a clique in $H \simeq G^2$. The converse is not true. To see this it is sufficient to note that if $C_3 \simeq G^2$ then $G$ can be $C_3$ or $P_3$. So the algorithm builds $G^2$, and generates all cliques in $G^2$ and eliminates those cliques that are not a $D2CS$ in $G$.

Hence we have the following theorem.

Theorem 35. Let $k$ be the number of $D2CSs$ in a graph $G$. Then all $k$ $D2CSs$ in $G$ can be enumerated in time $O(kn^3/\log^2 n)$.

We next present a recursive algorithm for finding the size of a maximum $D2CS$ (MDS) of a graph $G$ i.e. we give an algorithm for finding the value of $\omega_2(G)$ which gives a lower bound on $\chi_{\Delta}(G)$.

Recursive Algorithm for Size of a Maximum $D2CS$ of a Graph

Brute force approach for MDS problem would check all subsets of $V(G)$, and return the largest subset whose induced subgraph in $G$ has diameter at most two. This naive algorithm runs in exponential time. Let $n$ and $m$ represent the number of vertices and edges in $G$.

Our algorithm for finding the size of MDS in $G$ exploits the fact that if a specific vertex $v \in V(G)$ is in MDS then the size of MDS is at most $N_2[v]$ – here $N_2[v]$ denotes the two neighbourhood of $v$, i.e. $N_2[v] = \{v' : d(v, v') \leq 2\}$, where $d(v, v) = 0$. Hence the size of MDS is equal to its size in the subgraph $G[N_2[v]]$ of $G$, in this case. If the vertex $v$ is not in MDS then the size of MDS is equal to its value in the subgraph $G \setminus \{v\}$. Our algorithm may be compared/contrasted with the recursive graph algorithm for finding the size of the maximum sized independent set given in [24].

Complexity Analysis. For a graph $G$ with $n$ vertices, let $F(G)$ be the computation effort spent in finding the size of a MDS of $G$. Then

$$F(G) = g(G) + F(G \setminus \{v\}) + F(G[N_2[v]])$$

where $g(G)$ is the time needed for checking whether $\text{diam}(G) \leq 2$. Considering different graphs of size $n$, let $f(n)$ be the maximum of $F(G)$ where $G$ has $n$ vertices i.e.,

$$f(n) = \max\{F(G) : |V(G)| = n\}.$$

We assume that the maximum degree $\Delta$ of $G$ is bounded by a constant factor i.e.,
\[ \Delta = cn \text{ for some constant } 0 < c < 1. \] From [4] it is clear that \( g(G) = O(n^3/\log^2 n). \) Then

\[ f(n) \leq O(n^3/\log^2 n) + f(n-1) + cn^2, \]

which implies \( f(n) = O(n^4/\log^2 n). \)

**Remark 36.** If the input graph \( G \) is disconnected, we work on each of its connected components; let \( n_i, m_i \) denote the number of vertices and edges of the \( i \)th connected component respectively. The computation of connected components takes \( O(m + n) \) time [8], while processing each of them takes \( O(n_i^4/\log^2 n_i) \) time. Since \( \sum_i n_i = n \), we have that \( O(n^4/\log^2 n) \) time suffices for finding the size of MDS in \( G \). Hence we have the following theorem.

**Theorem 37.** Let \( G \) be an undirected graph on \( n \) vertices whose degree is bounded. Then the size of MDS in \( G \) can be obtained in time \( O(n^4/\log^2 n) \).

**Linear-Time Algorithm for Enumerating Maximal D2CSs in a Strongly Chordal Graph**

We recall the following definition.

**Definition 8.** [10] A graph is strongly chordal if and only if it admits a s.e.o.

The following lemma gives the structure of the maximal D2CSs in a strongly chordal graph.

**Lemma 38.** Let \( G \) be a strongly chordal graph. Every maximal D2CS in \( G \) is of the form Type-1.

But the converse of the above lemma does not hold. The following result follows directly from the above lemma.

**Proposition 39.** Let \( G \) be a strongly chordal graph with \( n \) vertices. Let \( X \) be the maximum possible number of maximal D2CSs in \( G \). Then \( X \leq n \). The equality holds iff \( G \) has no edges.

Based on the above lemma we present an algorithm for enumerating maximal D2CSs in a strongly chordal graph which proves the following theorem.

**Theorem 40.** All maximal D2CSs in a strongly chordal graph on \( n \) vertices can be enumerated in time \( O(n) \).
Relation Between $\omega_2(G)$ and $\chi_r(G)$

Let the set $S$ be a D2CS of cardinality $l$ of $G$ such that for all $v \in S$ we have $d(v) \leq r$. Then

$$\chi_r(G) \geq l,$$

because all the vertices in $S$ must be colored distinctly as otherwise either (C1) or (C2) would be violated at one or more vertices of $S$. We therefore get the following upper bound on $\omega_2(G)$:

$$\omega_2(G) \leq \chi_\Delta(G).$$

RADIO LABELING OF GRAPHS

In 2001, Chartrand, Erwin, Zhang, and Harary explored the concept of Hale’s version of the radio channel assignment problem and introduced radio labeling of graphs. For an integer $k \leq \text{diam}(G)$, a radio $k$-labeling $f$ of $G$ is an assignment of non-negative integers, called labels to the vertices of $G$ such that if $u, v \in V(G)$ are distinct then

$$d(u, v) + |f(u) - f(v)| \geq k + 1$$

(1)

where $d(u, v)$ is the distance between $u$ and $v$. The radio $k$-labeling number $rc_k(f)$ of a radio $k$-labeling $f$ of $G$ is the maximum label assigned to a vertex of $G$. The radio $k$-chromatic number $rc_k(G)$ is $\min \{rc_k(f)\}$ over all radio $k$-labelings $f$ of $G$. A radio $k$-labeling $f$ of $G$ is a minimum radio $k$-labeling if $rc_k(f) = rc_k(G)$. Then $rc_1(G) = \chi(G)$. A radio $d$-labeling of $G$ is a radio labeling of $G$, and the radio $d$-chromatic number $rc_d(G)$ is the radio number $rn(G)$ and has been studied extensively in the past decade (see for example [9, 21, 17, 18, 20, 19]). A survey of known results on radio labeling can be found in [6].

Given a graph $G$ and two integers $k$ and $l$ the complexity of deciding whether $rc_k(G) \leq l$ is not yet known. Finding the radio number for a graph is an interesting, yet challenging task. So far, the value is known only for very limited families of graphs. Our objective is to investigate upper bounds for the radio number of some specific class of graphs.

The following theorem from [21] gives an upper bound for radio number of tree that is neither a star nor a path.

**Theorem.** Let $T_n$ be a tree of order $n \geq 5$ that is neither a star nor a path. Then for any integer $k \geq 2$, $rc_k(T_n) \geq (n - 1)(k - 1) - 1$.

The above theorem when applied to the radio number of a binomial tree $B_n$, a binary
Fibonacci tree $BFT_n$ of order $n$ ($n \geq 0$), Fibonacci tree $T_n$ of order $n$ and a uniform caterpillar $UC_n$ gives the following upper bounds.

(i) $rn(B_n) \leq 2(2^n - 1)(n - 1) - 1$.
(ii) $rn(BFT_n) \leq 2(n - 1)F_{n+3} - 4n + 3$.
(iii) $rn(T_n) \leq (F_{n+2} - 1)n - 2F_{n+2} + 1$.
(iv) $rn(UC_n) \leq (N - 1)(n - 2) - 1$, where $N = n + (n - 2)(\Delta - 2)$.

In the thesis improved upper bounds for the radio numbers of $B_n, T_n, BFT_n$ and $UC_n$ are given based on combinatorial ideas and case analysis. We also give upper bounds for radio number of strong product of paths based on the idea used in [17] to obtain the radio number of paths.

**Theorem 41.** For $n \geq 3$ let $B_n$ be a binomial tree of order $n$. Then
\[ rn(B_n) \leq 2(2^n - 1)(n - 1) - 1 - 3 \cdot 2^{n-3}. \]

**Lemma 42.** Let $BFT_n$ be a binary Fibonacci tree of order $n$ with $n \geq 2$. Let $D(n) = \sum_{i=1}^{n'-1} d(i, i+1)$, where $n' = F_{n+3} - 1$. Then
\[ D(n) = 2F_{n+5} - 5(n + 2). \]

**Theorem 43.** Let $BFT_n$ be a binary Fibonacci tree of order $n$ with $n \geq 3$ and let $n'$ be the total number of nodes in it. Then
\[ rn(BFT_n) \leq 2(n - 1)F_{n+3} - 2F_{n+4} + n + 10. \]

For integer $n > 0$, let $FT_n$ be a Fibonacci tree of order $n$ as defined in the previous section. We assume that the vertices of $FT_n$ are labeled level by level from 1 to $F_{n+2}$ sequentially from left to right starting with the root.

**Theorem 44.** Let $FT_n$ be a Fibonacci tree of order $n$ with $n \geq 3$ and $S = \sum_{i=1}^{F_{n+2}-1} d(i, i+1)$. Then
\[ rn(FT_n) \leq (F_{n+2} - 1)n - S. \]

We next recall the definition of uniform caterpillar.

**Definition 9.** A graph $G$ is called a caterpillar if $G$ is a tree such that the removal of the pendent vertices produces a path, the spine of the caterpillar. A uniform caterpillar, $UC_n$ is a caterpillar with only degree one and degree $\Delta$ vertices with $n - 2$ vertices on the spine.

**Theorem 45.** Let $UC_n$ be a uniform caterpillar with $n = \max\{i : P_i$ is a subgraph of $UC_n\}$ and $N = n + (n - 2)(\Delta - 2)$. Let $l = (n^2 - 4n - 2)/12$ and $p = (n^2 - 4n - 7)/2$. Then
\[ rn(UC_n) \leq \begin{cases} (N - 1)(n - 2) - l, & \text{if } n : \text{even and } \Delta \leq l, \\ (N - 1)(n - 2) - p, & \text{if } n \geq 7 \text{ and } n : \text{odd}. \end{cases} \]
Theorem 46. For \( n \geq 3 \), let \( P_n \otimes P_n \) be the strong product of graphs \( P_n \) and \( P_n \). Then

\[
\begin{align*}
n_n(P_n \otimes P_n) &\leq \begin{cases} 
4(k^2 + 1)(k + 1), & \text{if } n = 2k + 1, \\
2(2k^3 - k^2 + k + 1), & \text{if } n = 2k.
\end{cases}
\end{align*}
\]
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