NP-Completeness of Dominating Set

Let $G = (V, E)$ be a simple undirected graph. A dominating set in a graph $G$ is a subset of vertices $S \subseteq V$ such that each vertex in $V$ is either in $S$ or is adjacent to some vertex in $S$. That is, for every vertex $u \in V - S$, there exists a vertex $v \in S$ such that $uw \in E$. A dominating set is minimal if $S$ cannot be contracted further; that is, there exists no vertex $w \in S$ such that $S - \{w\}$ is also a dominating set in $G$. The problem of finding a minimal dominating set of minimum cardinality is a hard problem.

The DOMINATING SET (DS) decision problem is the following.

INSTANCE : Given a graph $G$ and an integer $k$.

QUESTION : Does $G$ have a dominating set of size at most $k$?

Theorem : DS is NP-complete.

Proof : For each vertex in the graph $G$, check whether that vertex is in the dominating set $S$ or adjacent to a vertex in $S$. That is, for each vertex we check every edge incident to it in $G$ to see if that edge connects the vertex to at least one vertex in $S$. If we ever find a vertex which is not in $S$ and is not adjacent to $S$, we reject. Otherwise we accept $S$ as the dominating set of the graph $G$. The overall algorithm runs in polynomial time, so DS is in NP.

The following reduction from 3SAT to DS will establish the NP-completeness of DS. Let $E$ be an instance of the 3SAT problem with $n$ variables $X_1, X_2, ..., X_n$ and $m$ clauses $C_1, C_2, ..., C_m$. Given this instance, we construct a graph $G$ with $3n + m$ vertices. Three vertices correspond to each variable; these are labelled $x_i, \overline{x}_i, y_i$. One vertex corresponds to each clause; these are labelled $c_i$. The graph $G$ has $3n + 3m$ edges. The three vertices corresponding to each variable are connected by an edge (that is, $x_i, \overline{x}_i, y_i$ forms a triangle). Each clause vertex is connected to its component terms (that is, the clause vertex $c_p$ corresponding to the clause $X_i + \overline{X}_j + X_k$ is connected by edges to the vertices $x_i, \overline{x}_j, x_k$).

For example, consider the 3SAT instance

$E = (X_1 + \overline{X}_2 + X_3)(X_2 + X_4 + \overline{X}_5)(X_3 + \overline{X}_4 + \overline{X}_6)(\overline{X}_5 + X_6 + X_7)$.

Then the graph $G$ is as given below.

It should be straightforward that this construction can be done in polynomial time in the length of the number of variables and number of clauses in $E$. We must show that this correctly reduces 3SAT to DS. That is :

$E$ is satisfiable if and only if $G$ has a dominating set of size at most $n$. 

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Assume $E$ is satisfiable. Then there exists an assignment of $\top$ or $\bot$ values to the variables such that all clauses evaluate to true. We will form a set $S$ which includes $x_i$ for all $X_i$ which are true in this assignment, and $\overline{x_i}$ for all $X_i$ which are false. This set contains exactly $n$ vertices. All the vertices which came from variables will be covered by this set, since one of $x_i, \overline{x_i}$ was selected for every $i$. Since the truth assignment makes every clause true, one of the component terms of each clause is true and therefore is in $S$. It follows that all the vertices are adjacent to vertices in $S$, and the set is a dominating set. Thus if $E$ is satisfiable then $G$ has a dominating set of size at most $n$.

Assume there exists a dominating set $S$ of size at most $n$. Since $y_i$ is either in $S$ or adjacent to a vertex of $S$ for all $i$, and $y_i$ is connected by edges only to $x_i$ and $\overline{x_i}$, it follows that for every $i$, either $x_i, \overline{x_i}$, or $y_i$ is in $S$. This already specifies $n$ vertices, so exactly one vertex for each variable is included in $S$. To dominate the $c$-vertices note that none of the $y$-vertices can be selected to be in $S$. We create a truth assignment as follows. $X_i$ will be assigned true if $x_i$ is in $S$. Otherwise $X_i$ will be assigned false. Consider clause $C_j$. The vertex $c_j$ was not in the dominating set (which only uses variable correlated vertices). So $c_j$ is adjacent to some $x_h$ or $\overline{x_h}$ in the dominating set. If $c_j$ is adjacent to $x_h$ in $S$, then since $X_h$ is set to true it follows that $c_j$ will be true. If $c_j$ is adjacent to $\overline{x_h}$ in $S$, then $x_h$ is not in $S$ and $X_h$ will be false, so $C_j$ will be true. It follows that this assignment is a solution for $E$ and $E$ is satisfiable. Thus if $G$ has a dominating set of size at most $n$ then $E$ is satisfiable.

Note that the set of $c$-vertices will yield a dominating set had we not included the $y$-vertices as in the construction. That would have given a dominating set in $G$ but not a natural truth assignment for $E$.

Thus there is a polynomial time reduction from 3SAT to DS. Since 3SAT is NP-complete, it follows that DS is also NP-complete. \qed

Reference