COMBINATORICS AND GRAPH THEORY

Induction (Z_+)

Basis : True for the first +ve integers

Induction Hypothesis : Inductive step $\forall_k (P(k) \to P(k+1))$ $k \in Z^+$

$\Downarrow$

Assumption that for any arbitrary integer k, if P(k) is true then P(k+1) is also true

Principle of Mathematical Induction

$P(1)$

$\forall k \in Z_+ \; P(k) \to P(k+1)$ — using Inference rule

$\forall_n P(n), n \in Z_+$

secret sharing
Fall of Dominos
Climbing an Infinite Ladder

Basis
Inductive step

Basis : $P(1)$

Inductive step : $P(k) \longrightarrow P(k+1) \; \forall \; k \in Z_+$

$P(n) \; \forall \; n \in Z_+$

Sum of first n positive integers

$$P(n): 1+2+3+\ldots n = \frac{n(n+1)}{2}$$

Basis : $P(1) \quad 1 = \frac{1(1+1)}{2}$

Inductive step : $P(k) \quad 1+2+\ldots+k = \frac{k(k+1)}{2}$

$$P(k+1): 1+2+\dots k+k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)((k+1)+1)}{2}$$

$\therefore$ $P(n)$ is true

i.e $P(n) = \frac{n(n+1)}{2}$

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

Q) Sum of first n odd integers is n^2

$P(n): 1+3+5+\dots(2n-1) = n^2$

Basis : $P(1): 1 = (1)^2 \Rightarrow$ True

Inductive step :

$P(k) = 1+3+5+\dots(2k-1) = k^2$

$P(k+1): 1+3+5+\dots+(2k-1)+2k+1$

$k^2+2k+1 \Rightarrow (k+1)^2$

$\therefore$ $P(n)$ is True

$1+3+\dots+(2n-1) = n^2$

Q) $P(n): n < 2^n$

Basis : $P(1): 1 < 2^1 \Rightarrow$ True

Inductive step : $P(k)$ $\cancel{x}$: $k < 2^k$

Adding 1 on both sides

$P(k+1): k+1 < 2^k+1 \leq 2^k+2^k = 2.2^k$

as $1 < 2^k$

$k+1 < 2^{k+1}$

$\therefore$ $P(n)$ is true, $n < 2^n$

Harmonic series :

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\dots \frac{1}{n}$$

prove that $H_{2^n} \geq 1+ n/2$, n is a non-negative integer

$P(n)$: $H_{2^n} \geq 1+ n/2$

Basis : $P(0)$: $H_{2^0} \geq 1+\frac{0}{2} \Rightarrow H_1 \geq 1$ (True)

Inductive step : $P(k)$: $H_{2^k} \geq 1+\frac{k}{2}$

$$H_{2^k} = 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{2^k} \geq 1+\frac{k}{2}$$

$$H_{2^{k+1}} = 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{2^k}+\frac{1}{2^k+1}+\frac{1}{2^k+2}+\dots\frac{1}{2^{k+1}}$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{2^k}+\underbrace{\frac{1}{2^k+1}+\frac{1}{2^k+2}+\dots\frac{1}{2^k+2^k}}_{\text{Total of } 2^k \text{ terms}}$$

$$\geq 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{2^k}+\frac{1}{2^k+2^k}+\frac{1}{2^k+2^k}+\dots\frac{1}{2^k+2^k}$$

$$\geq 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{2^k}+2^k\left[\frac{1}{2^k+2^k}\right]$$

$$\geq 1+\frac{k}{2}+\frac{1}{2}$$

$$\geq 1+\frac{(k+1)}{2}$$

$\therefore$ $P(n)$ is True,

$$H_{2^n} = 1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{2^n} \geq 1+\frac{n}{2}.$$

Q) Prove that $1^2+3^2+5^2+\ldots(2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ where n is a non-negative integer

$P(n): 1^2+3^2+5^2+\ldots(2n+1)^2 = (n+1)(2n+1)(2n+3)/3$

Basis: $P(0) = (1)^2 = (1)(1)(3)/3 \Rightarrow$ True

Inductive step:

$P(k) = 1^2+3^2+5^2+\ldots(2k+1)^2 = (k+1)(2k+1)(2k+3)/3$

$$P(k+1) = 1^2+3^2+5^2+\ldots(2k+1)^2+(2k+3)^2$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$$

$$= (2k+3)\left[\frac{(k+1)(2k+1)}{3} + \cancel{1}\ 2k+3\right]$$

$$= (2k+3)\left[\frac{2k^2+3k+1+6k+9}{3}\right]$$

$$= (2k+3)\left[\frac{2k^2+9k+10}{3}\right]$$

$2k^2+9k+10$

$2k^2+4k+5k+10$

$(2k+5)(k+2)$

$$= \frac{(2k+3)(k+2)(2k+5)}{3}$$

$$= \frac{((k+1)+1)(2(k+1)+1)(2(k+1)+3)}{3}$$

$\therefore$ P(n) is true

i.e

$$P(n) = 1^2 + 3^2 + 5^2 + \dots (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

Q) Prove that $3 + 3\cdot5 + 3\cdot5^2 + \dots 3\cdot5^n = \frac{3(5^{n+1}-1)}{4}$

whenever n is a non-negative integer

$$P(n): 3 + 3\cdot5 + 3\cdot5^2 + \dots 3\cdot5^n = \frac{3(5^{n+1}-1)}{4}$$

Basis : $P(0) = 3\cdot5^0 = \frac{3(5-1)}{4} \Rightarrow$ True

Inductive step:

$$P(k) = 3 + \cancel{3}\, 3\cdot5 + 3\cdot5^2 + \dots 3\cdot5^{k} = \frac{3(5^{k+1}-1)}{4}$$

$$P(k+1) = 3 + 3\cdot5 + 3\cdot5^2 + \dots 3\cdot5^{k+1}$$

$$= \frac{3(5^{k+1}-1)}{4} + 3\cdot5^{k+1}$$

$$= 3\cdot\left[\frac{5^{k+1}-1}{4} + 5^{k+1}\right]$$

$$= 3\cdot\left[\frac{5^{k+2}-1}{4}\right]$$

$$= 3\cdot\left[\frac{5^{(k+1)+1}-1}{4}\right]$$

$\therefore$ P(n) is true

i.e $P(n): 3 + 3\cdot5 + 3\cdot5^2 + \dots 3\cdot5^n = \frac{3(5^{n+1}-1)}{4}$

Strong Induction (second principle of Mathematical Induction)

Basis : $P(1)$

$P(2)$

$\vdots$

$P(k)$

Assume $P(1), P(2) \ldots P(k)$ is true

$[P(1) \wedge P(2) \wedge P(3) \ldots \wedge P(k)] \rightarrow P(k+1)$

$P(n) \ \forall \ n$

For any $n > 1$, n can be either a prime no or written as the product of prime (prime factorisation)

Base : $P(2)$ is true

Inductive step : $P(j)$ is true for all $j \in Z_+$

$P(j+1) \rightarrow$ case 1: $j+1$ is prime

$P(j+1) \rightarrow$ case 2: $j+1$ is a composite Numbers.

case 2 : $j+1 = a * b$ both $a, b \leq j$

$2 \leq a, b \leq j$

Inductive Hypothesis : $a = P_1 \times P_2 \times \ldots P_k$

$b = q_1 \times q_2 \times \ldots q_{ki}$

$j+1 \Rightarrow P_1 \times P_2 \times \ldots \times P_k \times q_1 \times q_2 \times \ldots q_i$

Axious for the set of +ve integer.

1. The number 1 is a +ve integer
2. If n is a +ve integer, the successor of n, $n+1$ is also a +ve integer
3. Every +ve integer other than 1 is the successor of a +ve integer

4. Every non-empty subset of the set of +ve integer has a least element
(well ordering property)

If a is an integer and d is a +ve integer, then $\exists$ integers (unique) q, r with $0 \le r < d$ and

$a = dq + r$

$s = \{ a - dq \text{ for any } d, q \}$

well ordering property, s has a least element, say

$r = a - dq_0$

$r \geq 0$

$a = dq_0 + r$

$= d(q_0 + 1) + r_0$

$r_0 = r - d \quad r \not\geq d \Rightarrow r < d$

$a - d(q_0 + 1)$

which is the least element

Recursive Definition and structural Induction.

$Fact(n) = n \cdot fact(n-1)$

$fact(0) = 1$

$Fibo(n) = Fibo(n-1) + Fibo(n-2)$

$n \geq 2$

$Fibo(0) = 1$

$Fibo(1) = 1$

Recursively defined sets

Basis

Recursive step

Exclusion rule

Basis $3 \in S$

Recursive step if $x, y \in S \Rightarrow x + y \in S$

Ex. 2: Set of all strings $\Sigma = \{0, \dots 9\}$
alphabet

Σ^{+} Σ^{*} kleene star

(empty string is included)

empty string is not included

1. $\lambda \in \Sigma_1^{*}$ (empty string)
2. If $w \in \Sigma_1^{*}$, $x \in \Sigma_1$ then $wx \in \Sigma_1^{*}$

$\Sigma = \{a, b\}$ $\Sigma^{*} = \{\lambda, a, b, aa, ab, ba, bb, aab \dots\}$

Concatenation

1) If $w \in \Sigma^{*}$ then $w \cdot \lambda = w$
2) If $w_1, w_2 \in \Sigma_1^{*}$, $x \in \Sigma \Rightarrow w_1(w_2 x) = (w_1 w_2) x$

Ex 1: length of a string

1. $l(\lambda) = 0$
2. $l(wx) = l(w) + 1$ where $w \in \Sigma^{*}$, $x \in \Sigma$

Well formed formula (wff)

1. T, F, x (prepositional variable) is a wff
2. If x, y are wffs then $(\neg x)$, $(\neg y)$, $(x \wedge y)$, $(x \vee y)$, $(x \rightarrow y)$ are also wffs
3. Nothing else is wff

Task: p, q are proportional variables

$(p \neg \vee q)$, $(pq \wedge)$, $(\neg \vee pq)$ are not wff

Structural Induction

Basic step : $P(0)$

Recursive step : $P(k) \rightarrow P(k+1)$

Ex : S : $3 \in S$ prove that 's' represents the set of +ve integers divisibly by 3

$x, y \in S \rightarrow x+y \in S$

Soln: $A = \{3n / n \in N\}$; If $A \subseteq S$ $\forall$ $S \subseteq A$ then $A = S$

$A \subseteq S$

Basis : $n=1 \Rightarrow 3 \in A$ $\forall$ $3 \in S$ (as per definition)

Inductive step

$x, y \in A \Rightarrow x = 3k_1, y = 3k_2 \Rightarrow x+y = 3(k_1+k_2) \in A$

$\forall$ $\forall$

$x, y \in S$ $x+y \in S$

Basis : $3 = 3 \cdot 1 \Rightarrow 3 \in A$

Inductive step : $x, y \in S \Rightarrow x+y \in S \Rightarrow x+y = 3k \in A$

$\forall$

$x, y \in A$

Rooted Tree :

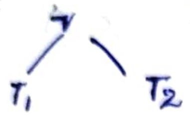
Basis : If 'r' is a single node, it is rooted tree.

Recursive step : If $T_1, T_2, \ldots T_k$ are some rooted trees, then the new graph with roots $r_1, r_2, \ldots r_k$, with a new root 'r' having connection to the existing roots $(r_1, r_2, \ldots r_k)$ is also a rooted tree

Full Binary tree : Either 0 or 2 children for each node

Basis : single vertex

Recursive step : T_1, T_2 are full binary trees. Connect T_1, T_2 as children of new root v to get full binary tree



Height - $h(T)$

Basis : $h(T) = 0$ for single vertex

Recursive step : T_1, T_2 are full binary trees ~~connect~~

$$h(T) = 1 + \max(h(T_1), h(T_2))$$

Nodes - $n(T)$

Basis : $n(T) = 1$ for single vertex

Recursive step : $n(T) = 1 + n(T_1) + n(T_2)$

Theorem : $n(T) \le 2^{h(T)+1} - 1$

Proof.

Basis : $n(T) = 1$, $h(T) = 0 \to 1 \le 2^{0+1} - 1$ (True)

Inductive step : ~~$\le +$~~

$$n(T) = 1 + n(T_1) + n(T_2)$$

$$\le \cancel{1} + 2^{h(T_1)+1} - \cancel{1} + 2^{h(T_2)+1} - 1$$

$$\le \cancel{2^{h(T_1) + h(T_2) + 2}} - 1$$

$$\le 2 \cdot 2^{\max(h(T_1), h(T_2)) + 1} - 1$$

$$\le 2 \cdot 2^{h(T)} - 1$$

$$\le 2^{h(T)+1} - 1$$

$n(T) \le 1 + n(T_1) + n(T_2)$

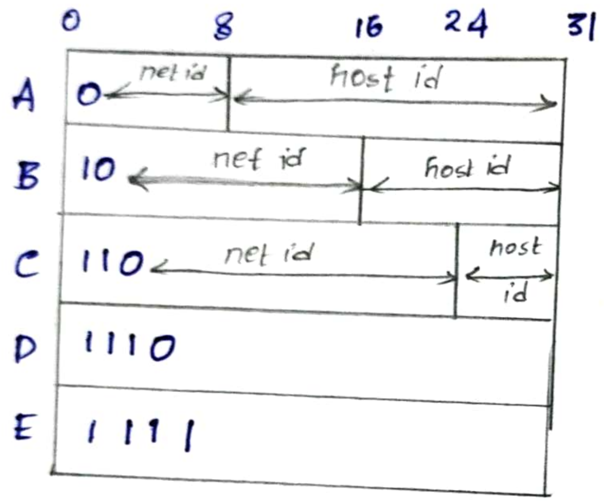
$\le 1 +$

```
n=0
for i=1 to k1
    for i=1 to k2
        for i=1 to k3
        .
        .
        for im=1 to km
            n=n+1
```

$n = k_1 \times k_2 \times \ldots \, k_m$

```
n=0
for i1=1 to k1
    n=n+1
for i2=1 to k2
    n=n+1
for i3=1 to k3
    n=n+1
.
.
for im=1 to km
    n=n+1
```

IPv4 address



01111111 ×

000.... 0

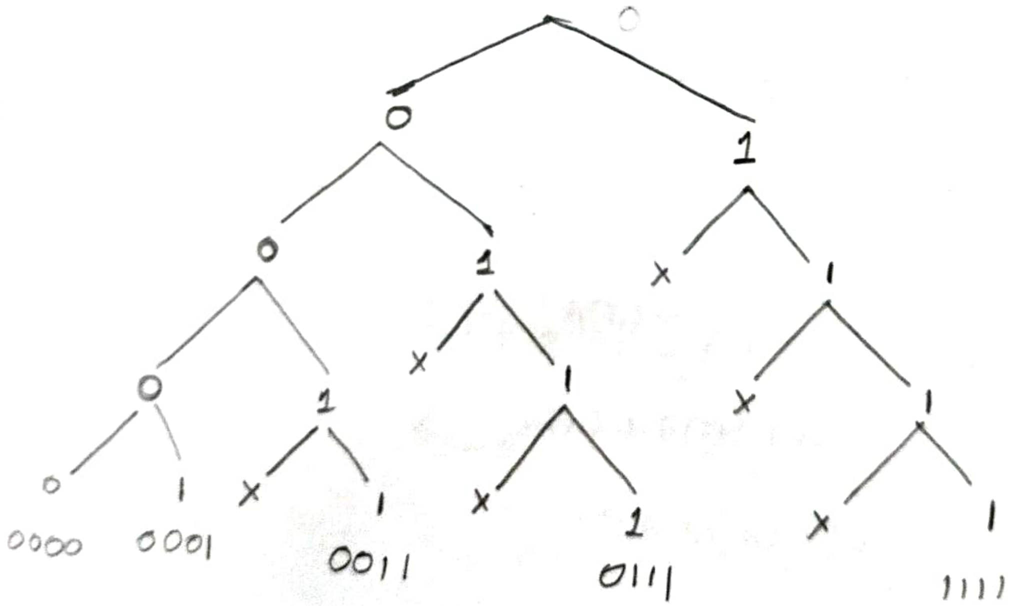
$A: (2^7 - 1)(2^{24} - 2)$

$B: 2^{14}(2^{16} - 2)$

$C: 2^{21}(2^8 - 2)$

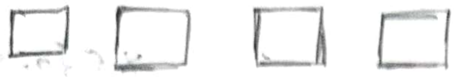
Tree diagram :

consider '10' is in valid



principle of inclusion & exclusion (or subtraction principle of counting)

Pegionhole principle: (Dirichlet drawer principle)



k-places & 'k+1' objects

For every integer 'n' there is a multiple of n, that has only 0s and 1s in its decimal expansion form.

consider $(n+1)$ numbers,

$1, 11, \ldots, \underbrace{11\ldots1}_{(n+1)\text{ digits}}$

Divide by n $\Rightarrow$ Remainder b/w 0 & $(n-1)$ $\Rightarrow n$ possibilities

* x, y have same remainder r (by pigeonhole principle)

Let $x > y$

$n > k$

If N objects are placed into k boxes, then there is atleast one box containing $\lceil \frac{n}{k} \rceil$ objects.

Proof : $\lceil \frac{n}{k} \rceil - 1 \rightarrow$ Maximum Number of objects in every box

$$\left(\lceil \tfrac{n}{k} \rceil - 1\right)k < \left(\left(\tfrac{N}{k}+1\right)-1\right)k = N$$

— contradiction

So, atleast one box should have more than $\left(\lceil \frac{n}{k} \rceil - 1\right)$ elements

$\Rightarrow \lceil \frac{N}{k} \rceil$ elements

$\lceil \frac{n}{k} \rceil \geq r$

$N = k(r-1)+1$ (Min N That satisfies the given inequality)

6 grades
atleast 10 students should get same garde

$\lceil \frac{N}{k} \rceil \geq r \Rightarrow N = k(r-1)+1$
$= 6(10-1)+1 = 55$

1.5 Billion

NXXX XXXX XXXX

$X \in \{0-9\}$
$N \in \{1-9\}$

$\left\lceil \frac{1.5 \text{ bill}}{10^8} \right\rceil = 15$

show that among any $n+1$ +ve integers $\leq 2n$ there must be an integer that ~~depends~~ divides one of the other integer.

$2^{k_i} \times q_i$ — q_i is odd $\Rightarrow n$ such odd Numbers $\leq 2n$

$\{ 2^{k_1} q_1, 2^{k_2} q_2, 2^{k_3} q_3 \, --- \, 2^{k_{n+1}} q_{n+1} \}$

so q_i, q_j Must be equal

$q_i = q_j = q$

$2^{k_i} q_i ; 2^{k_j} q_j$

$x = 2^{k_i} q$
$y = 2^{k_j} q$

$k_i > k_j$ $\quad \frac{x}{y} = 2^{k_i - k_j}$

Th: Every sequence of n^2+1 distinct real number contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Permutation & Combination

r- permutation
orderd arrangement of the object

$$^nP_r = \frac{n!}{(n-r)!}$$

$$= n(n-1)(n-2)\ldots(n-(r+1))$$

r- combination
unordered arrangement

$$^nC_r = \frac{n!}{(n-r)!\,r!} \quad \binom{n}{r}$$

Ex: $\underline{ABC}DEFGH$ No. of combinations if ABC is Together.

$6!\,3!$

$$^nC_r = \binom{n}{r}$$
↑
Binomial co-efficient

$^nC_r = {}^nC_{n-r}$

$^nP_r = {}^nC_r \times r!$

n bit string exactly r 1's ^nC_r

25 ECE 22 CSE

$\frac{22!}{3!\,19!}$

3	2
2	3

$$^{25}C_3 \times {}^{22}C_2 + {}^{25}C_2 \times {}^{22}C_3$$

$\frac{25!}{3!\,22!} \times \frac{23}{3}$

$^{25}C_2 \cdot {}^{22}C_2 \cdot \frac{^{43}C_1}{43}$

$\frac{25!}{2!\times 23!}$

$\frac{22!}{20!\,2!}$

$\frac{25!}{2!\,22!} \times \left(\frac{23}{3}\right)$

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + \ldots\ldots {}^nC_{n-1} x y^{n-1} + {}^nC_n y^n$$

co-efficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$

$$-{}^{25}C_{12}\, 2^{12}\, 3^{13}$$

$$\sum_{k=0}^{n} {}^nC_k = 2^n$$

$$\sum_{k=0}^{n} (-1)^k\, {}^nC_k = 0$$

n is +ve integer

Generalised permutation and combination.

1) r permutation of n ~~digits~~ objects with repetition $= n^r$

2) r combination of n objects with repetition $= {}^{n+r-1}C_r$

3) permutation with identical objects
(n_i objects to type i) $= \dfrac{n!}{n_1!\,n_2!\ldots n_k!}$

~~4)~~ Distinguishable objects into
distinguishable boxes
(n_i objects to put in x boxes)

4) n indistinguishable objects into k
indistinguishable boxes

$$a_1 + a_2 + \ldots a_k = n$$

$$a_1 \leq a_2 \leq a_3 \leq \ldots a_k$$

$\{a_1, a_2, a_3, \ldots a_k\}$ partition of n

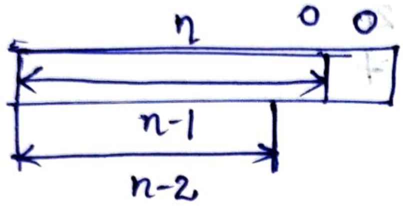
$$P_k(n)$$

Solving Recurrence

$\{a_n\}$ $\quad a_0, a_1, \ldots a_{n-1}$

Tower of Hanoi (exponential)

$$H_n = 2H_{n-1} + 1$$
$$= 2^n - 1$$



$a_0 = 0$ or 1

$a_1 = 01, 10, 11$

$a_2 = 010 \quad 100 \quad 110$

$\quad\;\; 011 \quad 101 \quad 111$

$$a_n = a_{n-2} + a_{n-1}$$

$A_1 A_2 \ldots A_k$

$A_{1\, n_1 \times n_2} A_{2\, n_2 \times n_3} \ldots$

$(A_1 A_2 \ldots)(\ldots A_n)$

$\quad (k) \qquad (n-k)$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k} \Rightarrow C_n = \frac{{}^{2n}C_n}{(n+1)}$$

$\{C_n\}$ is called the Catalan number

Linear homogeneous recurrence relation of degree k with constant co-efficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_k a_{n-k}$$

Let $a_n = r^n$ be a solution

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots c_k r^{n-k}$$

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$ characteristic equation

$k=2$

case 1a : $r^2 - c_1 r - c_2 = 0$

r_1, r_2 are the roots

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

case 1b : $r_1 = r_2 = r$

$a_n = \alpha_1 r^n + \alpha_2 n r^n$

$a_n = a_{n-1} + 2a_{n-2}$ $\quad a_0 = 2$, $a_1 = 7$

$r^2 - r - 2 = 0$ $\quad r = -1$ or 2

$a_n = \alpha_1 (-1)^n + \alpha_2 2^n$

$a_0 = 2 = \alpha_1 + \alpha_2$ $\quad \alpha_1 = -1$

$a_1 = 7 = -\alpha_1 + 2\alpha_2$ $\quad \alpha_2 = 3.$

$a_n = (-1)^{n+1} + 3(2)^n$

$a_n = 6a_{n-1} - 9a_{n-2}$ $\quad a_0 = 1$

$r^2 - 6r + 9 = 0$ $\quad a_1 = 6.$

$r = 3, 3$

$a_n = \alpha_1 3^n + (\alpha_2 n) 3^n$

$a_n = (\alpha_1 + \alpha_2 n) 3^n$ $\quad a_n = (n+1)3^n$

$f_n = f_{n-1} + f_{n-2}.$

$1 = \alpha_1 + \alpha_2$

$\cancel{6}_2 = \cancel{3}(\alpha_1 + 2\alpha_2)$

case 2a : $r_1, r_2, \ldots r_k$

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots \alpha_k r_k^n$

case 2b: root multiplicity

$r_1 - m_1$

$r_2 - m_2$

$\vdots$

$r_t \rightarrow m_t$

$$\begin{array}{ccccc} r_1, & r_2, & \ldots & , & r_t \\ \downarrow & & & & \downarrow \\ m_1 & & & & m_t \end{array}$$

$m_1 + m_2 + \ldots + m_t = k$

$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \ldots \alpha_{m_1} n^{m_1 - 1}) r_1^n$

$+ (\alpha_{2,0} + \alpha_{2,n} + \ldots \alpha_{2,1 m_2} n^{m_2 - 1}) r_2^n$

$+ (\alpha_{t,0} + \alpha_{t,1} n + \ldots \alpha_{t, m_t} n^{m_t - 1}) r_t^n$

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

$r^3 - 6r^2 + 11r - 6 = 0$ $\quad a_0 = 2$

$r = 1, 2, 3$ $\quad a_1 = 5$

$a_2 = 15$

$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$

$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 3^n$ $\quad a_0 = 2$

$\alpha_1 = 1$

$\alpha_2 = -1$

$\alpha_3 = 2$

$a_n = 1^n + (-1)2^n + 2 \cdot 3^n$

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ $\quad a_0 = 1$

$(r+1)^3 = 0$ $\quad a_1 = -2$

$r = -1$ with $m_1 = 3$ $\quad a_2 = -1$

$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n$

$\alpha_{1,0} = 1$

$\alpha_{1,1} = 3$

$\alpha_{1,2} = -2$

$a_n = (1 + 3n - 2n^2)(-1)^n$

$f_n = f_{n-1} + f_{n-2}$.

Linear non-homogeneous recurrence relation with constant co-efficients

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots \; c_k a_{n-k} + F(n)$ — (1)

$\{a_n^p\}$ particular solution for (1) then the actual solution $\{a_n^p + a_n^h\}$

a_n^h is the solⁿ of the associated homogeneous recurrence relation

$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0) s^n$

b_i, s are some real number.

(i) s is not a root of the linear homogeneous recurrence relation.

$$a_n^p = (P_t n^t + P_{t-1} n^{t-1} + \ldots P_1 n + P_0) s^n$$

(ii) s is a root of linear homogeneous recurrence relation with multiplicity m

$$a_n^p = n^m (P_t n^t + P_{t-1} n^{t-1} + \ldots P_1 n + P_0) s^n$$

$a_n = 3a_{n-1} + 2n$

$a_n^h = \alpha 3^n$ $\quad a_n = 3a_{n-1}$

$r = 3$

$F(n) = 2n$

$= (2n + 0) 1^n$

$a_n^p = (P_1 n + P_0) 1^n$

$P_1 n + P_0 = 3(P_1(n-1) + P_0) + 2n$

$P_1 n + P_0 = 3P_1 n - 3P_1 + 3P_0 + 2n$

$2P_1 n + 2P_0 - 3P_1 = 0$

$+2n$

$2P_1 + 2 = 0$

$\cancel{-3P_0 - 8}$ $\quad P_1 = -1$

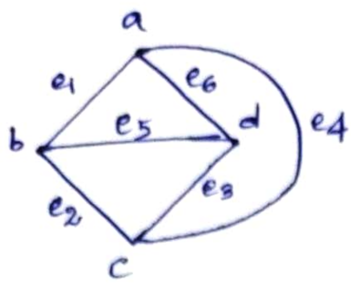
$P_0 = -3/2$

$a_n^p = (-n - 3/2)$

$a_n = \alpha 3^n - n - 3/2$

$\alpha 3^n - n - 3/2 = 3 \cdot \left[\alpha 3^{n-1} - (n-1) - 3/2 \right] + 2n$

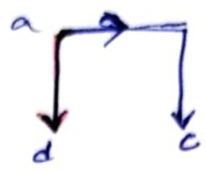
GRAPH THEORY



R symmetric relation

$e_1 = aRb$ & bRa

Non-directional edge



aRb
bRc
aRd

v - Number of vertices
e - Number of edges

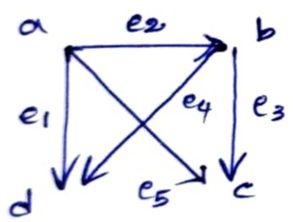
Incidence Matrix : $M(G) = [m_{ij}]_{v \times e}$

Undirected

$m_{ij} = 1$ if edge e_j is incident on v_i
0 if otherwise

$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline a & 1 & 0 & 0 & 1 & 0 & 1 \\ b & 1 & 1 & 0 & 0 & 1 & 0 \\ c & 0 & 1 & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

Directed.



$$\begin{array}{c|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ \hline a & 1 & 1 & 0 & 0 & 1 \\ b & 0 & -1 & 1 & 1 & 0 \\ c & 0 & 0 & -1 & 0 & -1 \\ d & -1 & 0 & 0 & -1 & 0 \end{array}$$

Disconnected graph

$$M(G) = \begin{array}{cc} g_1 & g_2 \end{array} \begin{bmatrix} M(g_1) & 0 \\ 0 & M(g_2) \end{bmatrix}$$

degree of a vertex - Total no of 1's and -1's in that row

Adjacency Matrix :

$A(G) = [a_{ij}]_{v \times v}$

Undirected :

$a_{ij} = 1$ if there exists edge b/w v_i & v_j
0 if otherwise

Directed

$a_{ij} = 1$ if $v_i \rightarrow v_j$
0 otherwise

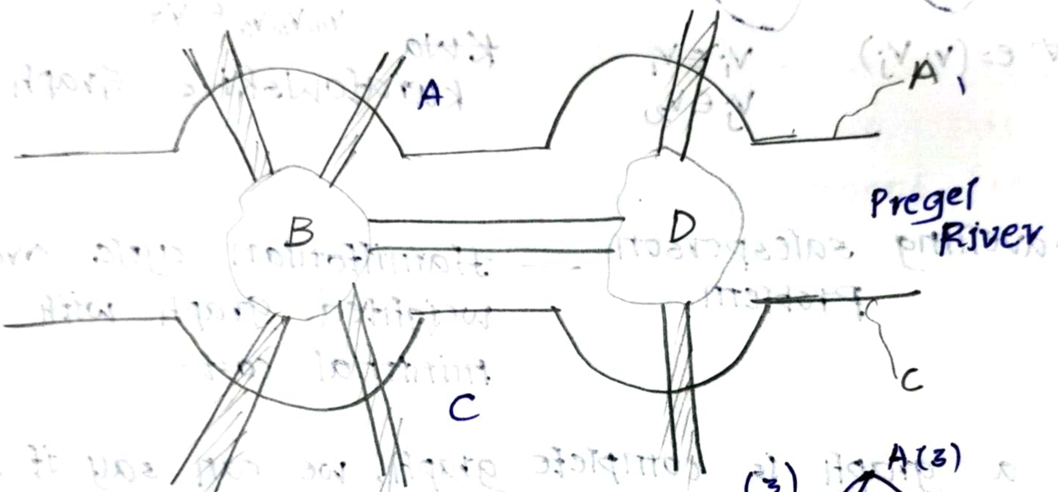
Hamiltonian Circuit / cycle

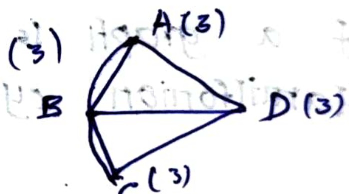
- Traverse each vertex exactly once and come back to first vertex.
- except first vertex, each vertex is traversed once only

Euler path / circuit

each edge is passed exactly once.

Konigsberg Bridge Problem Euler 1736





Theorm: Given connected graph it is having Euler path if and only if all the vertices are of even degree.

cor: If a graph has exactly 2 vertices of odd degree, it can have euler path starting from one vertex & ending at another.

Th 1: No. of vertices of odd degree in a graph is even.

Th 2: If exactly two vertices of odd degree are present in a graph, there must exist a path connecting them.

Th 3: A simple graph with n vertices of k components have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Simple graph —— no loops, no $\parallel^{el}$ edges

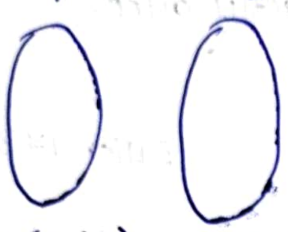
Regular graph — All vertices have same degree.

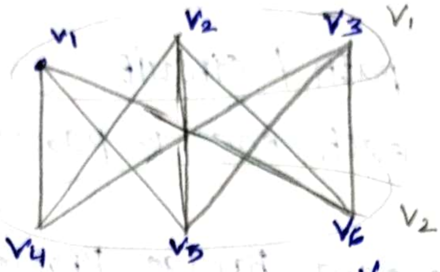
Complete graph  K_4 $\forall v_i, v_j \ (i \neq j) \ v_i \to v_j$

Null graph —— only isolated vertex, No edges

Bipartite graph - There must be a connectⁿ b/w 2 set of vertices, but not b/w them.

every pair of vertices are connected.





$\forall e = (v_i, v_j)$ $v_i \in V_1$, $v_j \in V_2$

* All complete graphs are regular, degree 'n-1'

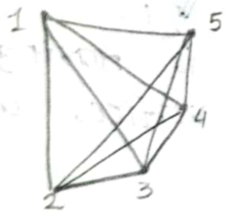
$K_{3,3}$ $v_1, v_2, v_3 \in V_1$
$v_4, v_5, v_6 \in V_2$

~~Kwa~~ Kuratowski's Graph

Travelling salesperson Problem — Hamiltonian cycle over a weighted graph with minimal cost.

If a graph is complete graph, we can say it has Hamiltonion cycle

(Because A→B & B→A are included)



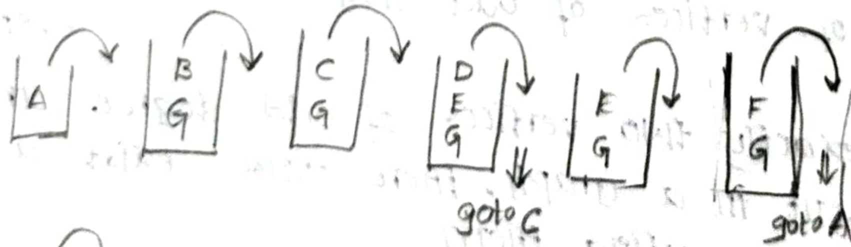
$\frac{(n-1)!}{2}$ Total number of Hamiltonian cycles

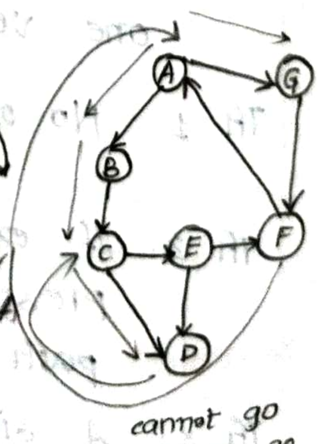
$\frac{n-1}{2}$ => Number of edge disjoint circuits.

GRAPH TRAVERSAL (Back Tracking)

DFS: Depth first search - Uses Backtrack

BFS: Breadth first search





cannot go anywhere go back to C

A
B G
G C
C F
F E D
E D
D

ABGCFED

queue.

Havel - Hakimi Algorithm

solves graph realization problem → represents degree of each of the vertices

Graphic seq$(s, t_1, t_2, \dots t_s, d_1, d_2, \dots d_n)$
↳ each element represents degree of a vertex.

Given a seq( ) be a finite list of non-negative integers which is non-increasing.

Let $A' = (t_1 - 1, t_2 - 1, \dots t_s - 1, \lambda_1, \lambda_2, \dots \lambda_n)$ be another list of non-negative integers that can be arranged into non-increasing order.

To generate A' => Remove highest element (i.e 1^{st} element) for A and decrease 1^{st} s elements by 1

Q) $A = (6, \overbrace{3, 3, 3, 3, 2, 2}^{\text{subtract 1}}, 2, 2, 1, 1)$

$A_1 = (\overset{3-1}{2}, 2, 2, 2, 1, 1, 2, 2, 1, 1)$

↳ (Not in non-increasing

$A_1 = (\cancel{2}, \overbrace{2, 2}, 2, 2, 2, 1, 1, 1, 1)$

$A_2 = (2-1, 2-1, 2, 2, 2, 1, 1, 1, 1)$

$= (1, 1, 2, 2, 2, 1, 1, 1, 1) \Rightarrow (2, 2, 2, 1, 1, 1, 1, 1, 1)$

$A_3 = (1, 1, 1, 1, 1, 1, 1, 1)$

A_4

$A_7 = (0, 0, 0, 0)$

↙ graphic sequence

since all values are 0, all 4 vertices have degree 0, therefore the given sequence represents a graph.

★ essentially we're removing the vertex with highest degree and checking if it is still a graph.

{3 3 2 2 2}

{2 1 1 2}

{2 2 1 1}

{1, 0, 1}

{1, 1, 0}

{0, 0}

$A = (6, 5, 5, 4, 3, 2, 1)$

$A_1 = (4, 4, 3, 2, 1, 0)$

$A_2 = (3, 2, 1, 0, 0)$

$A_3 = (1, 0, 0, 0)$

$A_4 = (-1, 0, 0)$

↓ Not a graphic sequence

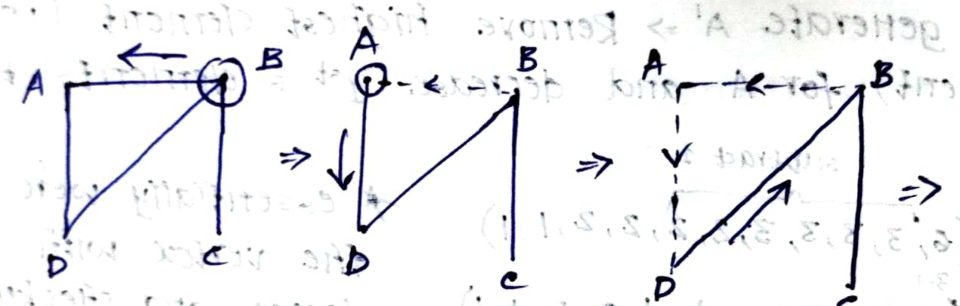
Fleury's Algorithm

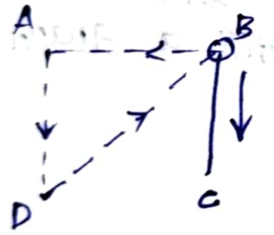
(To check for euler path / circuit)

1) The graph has to be connected
If the graph is disconnected, exist with "Not a Euler Graph".

2) If graph is having every vertex of even degree or 2 vertices of odd degree, goto step 3, otherwise exit ~~is~~ with "Not a Euler Graph".

3) start with one of the two vertices of odd degree (or any vertex of even degree) for even circuit





B → A → D → B → C

4) choose an edge whose deletion does not disconnect the graph (If no such edge exists, chose the remaining edges)

5) Move to the end point of the edge & delete the edge from the graph.

6) Repeat step 4,5 until no edges are left

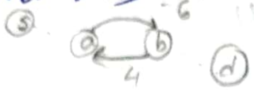
7) Traversed edges form an Euler cycle if all the vertices of even degree else it is Euler Path.

Shortest Path Algorithm

source s
destination d
sum of ~~ee~~ weight of edges from source to destination must be Min

weighted graph
V, E
$w : E \to \mathbb{R}$

1) Dijkstra's Algorithm
2) Bellman ford Algorithm — Dynamic App (exhaustive) in servers



Dijkstra's Algorithm (Greedy App)

step 1 : $P = \{\overset{s}{\cancel{\emptyset}}\}$ $T = V \setminus \{s\}$

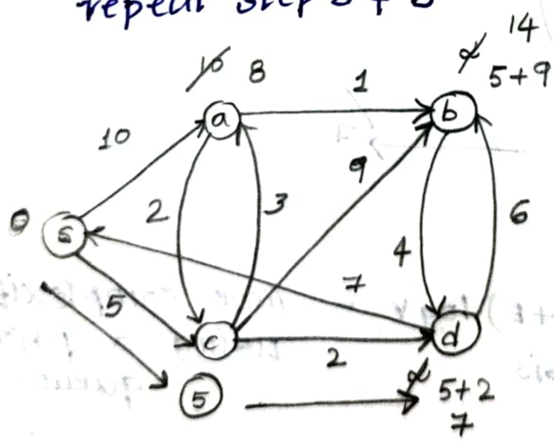
$\forall\ v \in T$ $l(v) = W(s, v)$

step 2 : select the vertex with min l, Let that vertex is a

step 3 : If a is destination, stop, otherwise set $P' = P \cup \{a\}$, $T' = T \setminus \{a\}$

$$\forall\ v \in T', l'(v) = \min(l(v), l(a) + W(a, v))$$

step 4 : set $T = T'$
$P = P'$
repeat step 2 & 3



It 1 : $P = \{s\}$
$T = \{a, b, c, d\}$

It 2 : $P = \{s, c\}$
$T = \{a, b, d\}$
Min dis $s \to d$ is 7

Bellman ford: (dynamic programming)

- Allow negative weight
- detect negative cycle

exhaustic app (exponential time)

1) BF (G, w, s)

$\forall \; v \in V[G]$

$d[v] \leftarrow \infty$

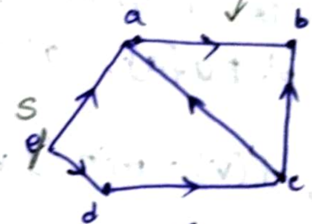
$\pi[v] \leftarrow NIL$

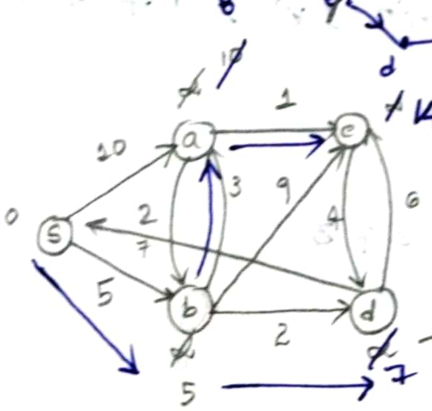
$d[s] \leftarrow 0$

Shortest path problem

- single source single destination shortest path
- single source All destination shortest path
- All source single destination shortest path

Swap both src, dest^n and reverse the direction.





(we cannot reach a directly)

Backtracking is allowed

O (V+E) log V — Time complexity using a priority queue

Amortise Analysis

2) For $i = 1$ to $|V[G]| - 1$

do for each $e = \overset{u,v}{\cancel{(v,u)}} \in E[G]$

if $d[v] > d[u] + w(u,v)$

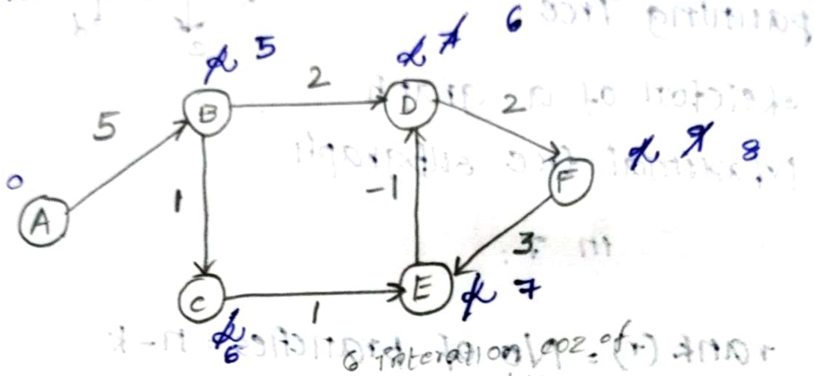
$d[v] \leftarrow d[u] + w(u,v)$, $\pi[v] \leftarrow u$

3) For each edge $(u,v) \in E[G]$

if $d[v] > d[u] + w(u,v)$

return False

else return True



6 iteration

6 edges

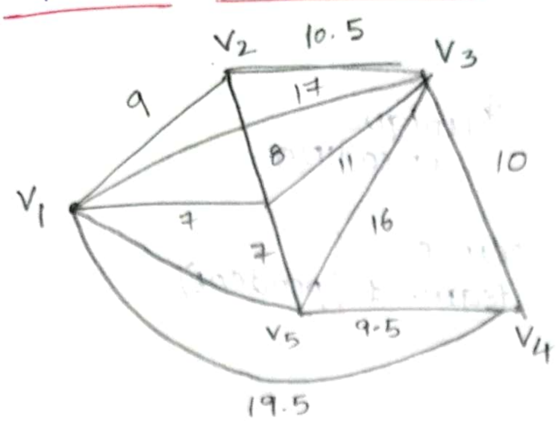
i	A	B	C	D	E	F
0	0	∞	∞	∞	∞	∞
1	0	5	∞	∞	∞	∞
2	0	5	6	7	∞	∞
3	0	5	6	7	7	9
4	0	5	6	6	7	9
5	0	5	6	6	7	8

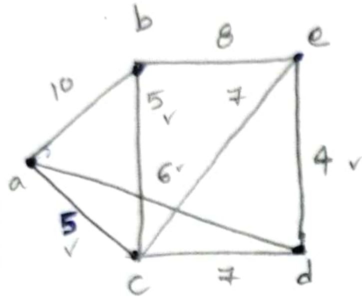
d

π

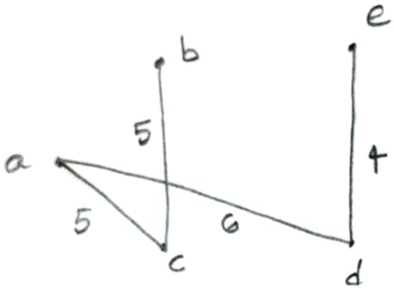
for 3 in an edge, there will be 6th iteration

Prims Algorithm





	a	b	c	d	e
a	-	10	5	6	∞
b	10	-	5	∞	8
c	5	5	-	7	7
d	6	∞	7	-	4
e	∞	8	7	4	-

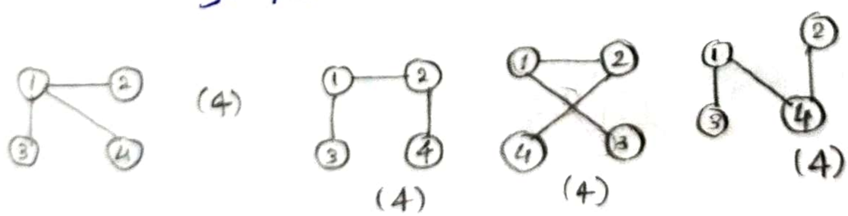


Carl Borchadt

Cayley's Formula

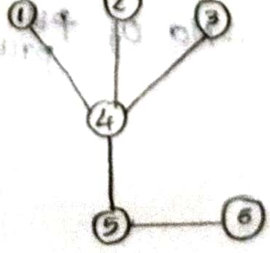
labelled Trees : n^{n-2} (connected graph) n_vertex

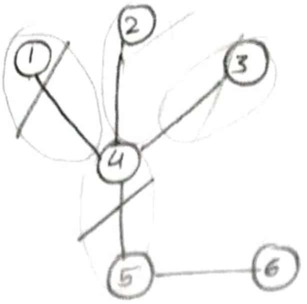
k-component graph : $T_{n,k} = k \cdot n^{n-k-1}$



spanning Trees for n-vertex complete graph $\Rightarrow n^{n-2}$

Prufer's Encoding





{4, 4, 4, 5}

↓ unique encoding

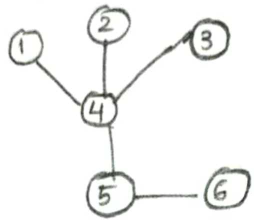
unique prefer's encoding for a Particular tree.

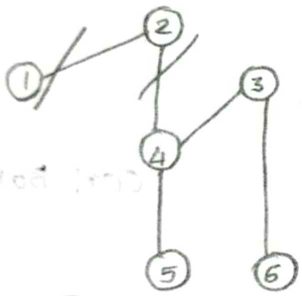
vertices of degree 1 (pendent)

{4, 4, 4, 5}

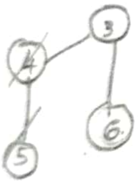
4 of degree 4

5 of degree 2





{2, 4, 4, 3}



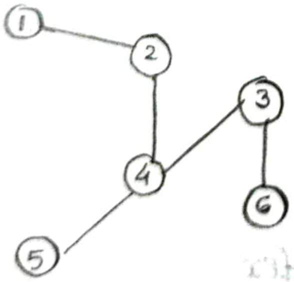
{2, 4, 4, 3}

2 — degree - 2

4 — degree - 3

3 — degree - 2.

1, 5, 6 degree - 1



For a n_vertex tree, prufer sequence will have n-2 elements.

∴ No of ~~PH~~ prufer sequence = n^{n-2}

$\begin{cases} f : T_n \to P_n & f \circ f^{-1} = ID \\ f^{-1} : P_n \to T_n \end{cases}$

Bijective Mapping

no of trees with 'n' nodes.

all possible Prufer sequences

$|T_n| = |P_n| = n^{n-2}$

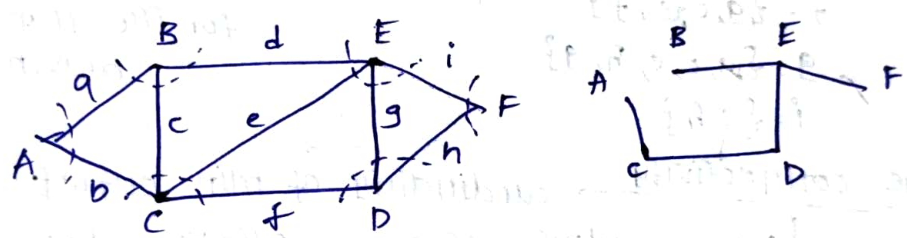
This is possible in complete graph where there is a connection b/w every vertex

represents no of spanning trees in a complete tree.

Cut-sets

→ Minimum number of edges whose removal results in a disconnected graph

→ Also known as cocycle / Min cutset / cutset



$Q \to$ a branch selected from every spanning Tree G

$Q \to$ set

$G - Q \to$ disconnected graph

Th. In a connected graph, the minimal set of edges containing on branch from every spanning tree is a cutset.

Fundamental circuit : wrt T (spanning tree) is obtained by adding a chord to T

Fundamental cutset wrt T : Removing one branch and all other chords related to that branch makes it disconnected.

Th :

wrt a spanning tree T_1, a chord c that determines a fundamental circuit T appears in every fundamental cutset associated with branches of T and nowhere else

Th:

w.r.t T, a branch that determines the fundamental cutset S is contained in every fundamental circuit associated with the chords in S, and in no other

T: {b, f, d, g, i}

Fundamental circuit

a= {a, b, f, ~~g~~ d, g}
c: {c, f, g, d}
e: {e, f, g}
h: {h, g, i}

Fundamental cutsets

b: {a, b}
d: {a, c, d}
f: {a, c, e, f}
g: {a, c, e, h, g}
i: {i, h}

edge connectivity ⇒ min cutset cardinality

for the graph drawn → 2.

Edge connectivity → cardinality of min cutset of a graph

↳ for pendent vertex ⇒ 1 (Having edge

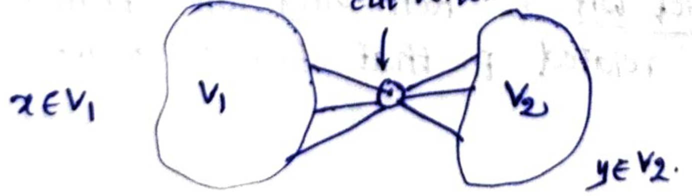
Th: Edge connectivity cannot exceed the degree of the vertex with least degree.

Vertex connectivity → Remove min no. of vertex to make the remaining graph as disconnected.

Th:

vertex connectivity cannot exceed edge connectivity of the graph.

VC ≤ EC



separability : If vertex connectivity is 1

cut-vertex: The special vertex which makes the graph disconnected / separable.

Th: A vertex z is a cut vertex iff $\exists$ two sets of vertices V_1 and V_2 in G for any $x \in V_1$ and $y \in V_2$, the connection b/w x and y goes through z

Th: The maximum vertex connectivity one can achieve with G (n vertices & e edges) in (average degree) $\left\lfloor \frac{2e}{n} \right\rfloor$

$$V.C \le E.C \le \frac{2e}{n}$$

$$\text{Max } V.C = \left\lfloor \frac{2e}{n} \right\rfloor$$

k-connectivity

k connected graph

Th: A connected graph is k-connected iff every pair of vertices is connected with k or more which do not intersect & there is atleast one pair of vertices which are connected with exactly k-non intersecting path

You cannot directly remove 1 vertex to make it disconnected.

Th: A connected graph has edge connectivity as iff every pair of vertices is connected by k or more edge-disjointed paths & there is by atleast on pair of vertices which are joined exactly by k edge-disjoint paths.

Non-separability If every pair of vertices can be placed inside a cycle / circuit

You have to remove atleast 2 vertices.

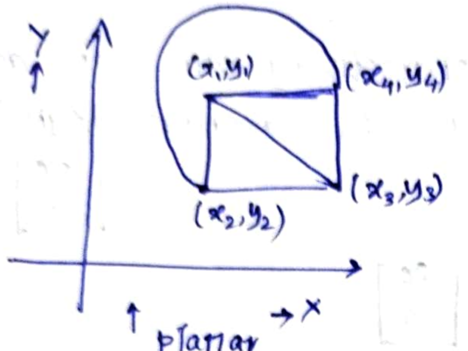
Planar Graph & its duals.

Embedding

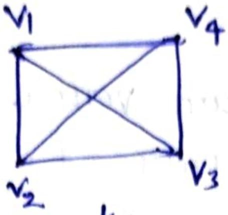
A drawing of a geometric representation of a graph on a surface s.T no edge intersect.

Planar Graph

→ x-y plane embedding of a graph



planar (should not intersect)

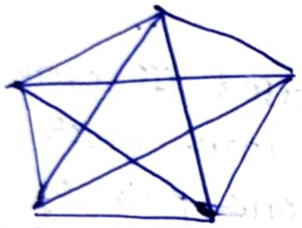


K_4 Not planar

kuratowski

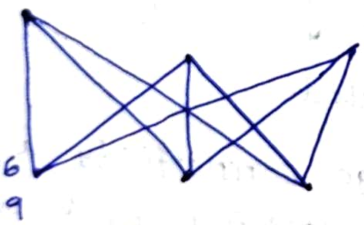
face

$K_{3,3}$ is complete bipartite graph



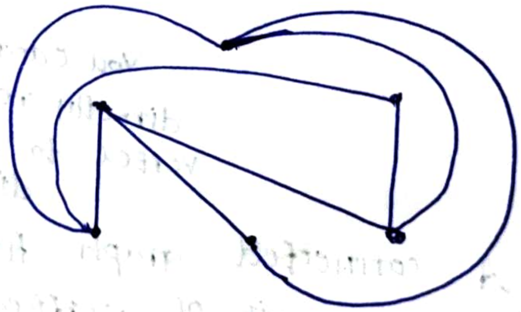
$n=5$ K_5
$e=10$

cannot be planar



$n=6$
$e=9$

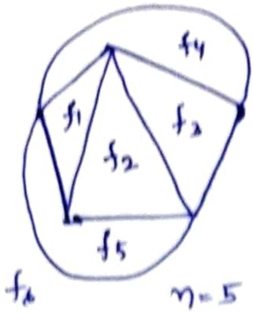
$K_{3,3}$.
Not a planar
(If we remove one edge or vertex, it becomes planar)



Min no of vertices to create a non planar graphs → K_5

~~Max~~ Min no of edges to create a planar graph → $K_{3,3}$.

concept of face only exist for planar graph.



degree of a face
= # edge covering the face.

$\sum_i d(f_i) = 2e$

$n = 5$
$e = 9$
$f = 6$

Euler's Theorm

$f = e - n + 2$

$n + f - e = 2$ → eulers characteristics

Necessary condition

simple connected graph $e \geq \frac{3f}{2}$

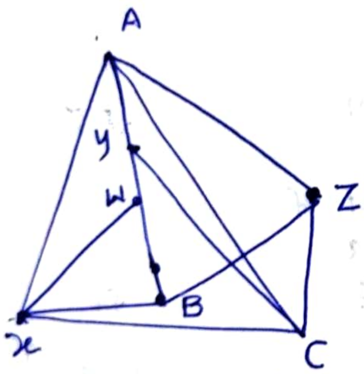
$e \leq 3n - 6$.

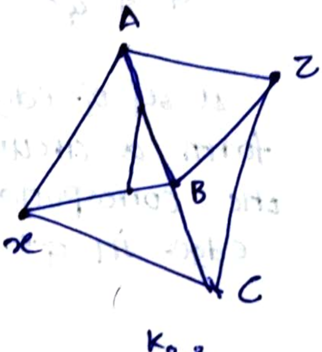
Bipartite graph $e \geq 3f/2$

$e \leq 2n - 4$

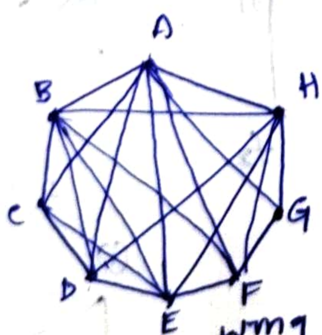
Th: A necessary & sufficient condition for a graph to be a planar graph is it does not contain either of K_5 or $K_{3,3}$ or any graph homomorphic to either of them

homomorphic: A graph where edges in series are removed / added.





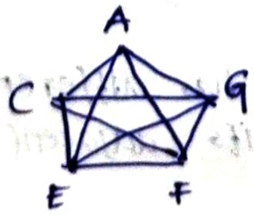
$K_{3,3}$.

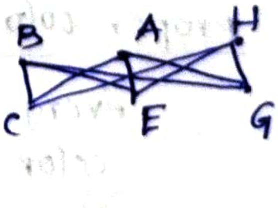


wrng diagram.

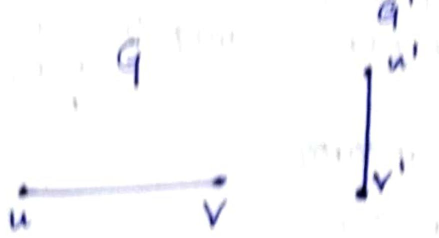
$e \leq 3n - 6$
$3 \times 8 - 6$
18.

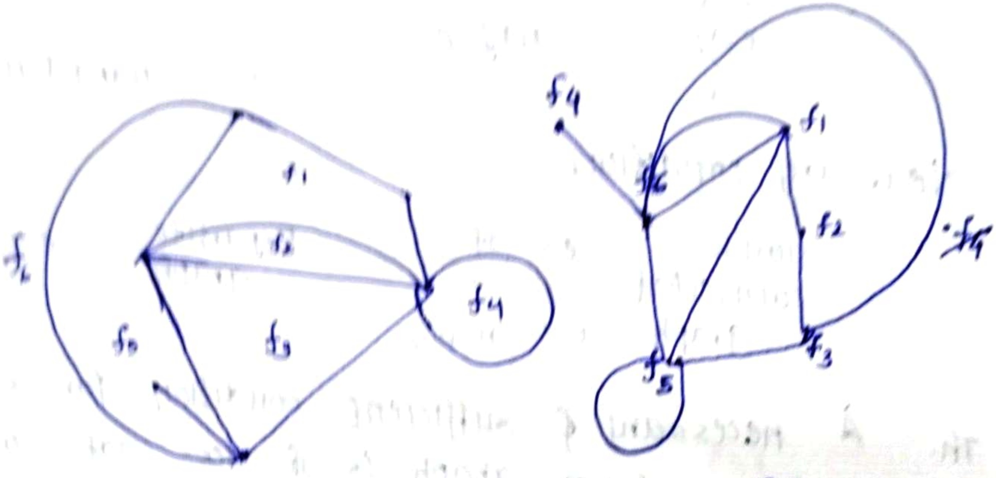
show that G includes Both K_5 & $K_{3,3}$.





Dual graph





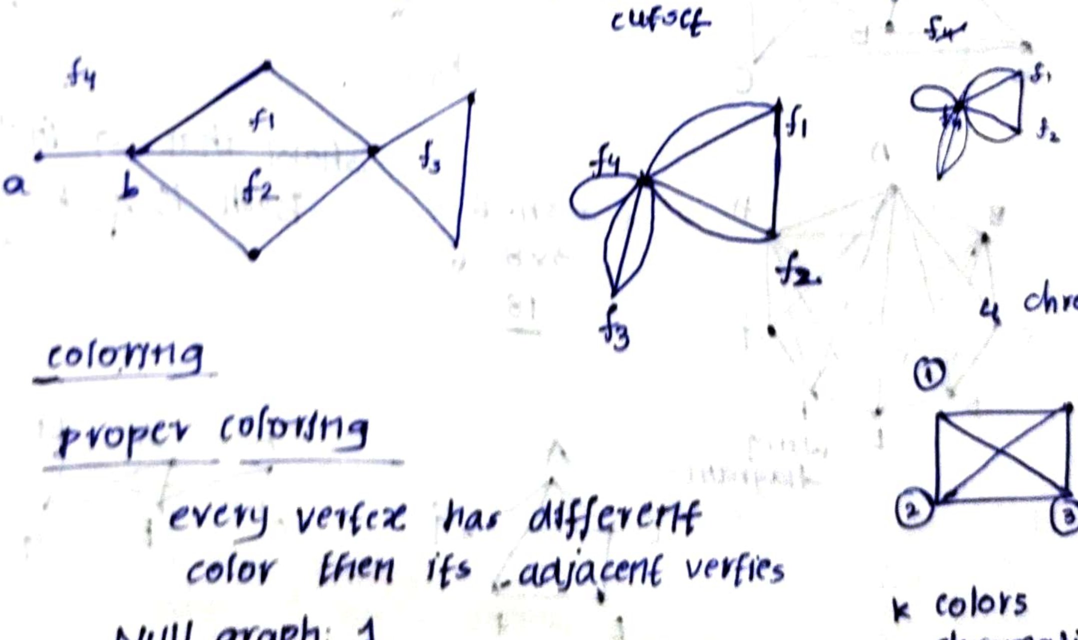
Pendent → self loop
Self loop → pendent

No of Branches in a Tree – Rank

Th: one-to-one rel^n b/w the edges in G & G*

G	G*
n	$n^* = f$
e	$e^* = e$
f	$f^* = n$
r	$r^* = \mu$
μ	$\mu^* = r$

a set of edges in G form a circuit iff the corresponding set of edges in G* form a cutset



coloring

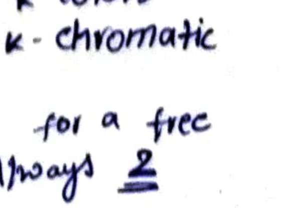
proper coloring

every vertex has different color then its adjacent vertices

Null graph: 1

Atleast one edge → Atleast 2

complete graph — n Color



k colors
k - chromatic

for a tree
Always 2

Kempe [1879] → Five Color Theorem [1890., Heawood]

$$e \le 3n-6$$

$$\frac{2e}{n} \le \frac{6n-12}{n} = 6 - \frac{12}{n}$$

< 6 $\quad \exists$ v which has degree 5 or less

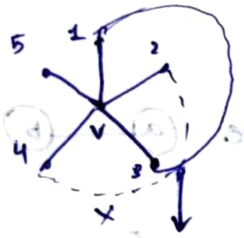
Th: The chromatic number of a planar graph of n vertices is at max 5

Basis: n = 1 , 5 ⇒ Basis is true

Inductive step: Assume, for a planar graph with n = k - 1 vertices, 5 colors are sufficient for proper coloring [I.H]

Take a graph with k vertices, Reverse v → Remaining graph can be properly coloured.

If v has degree ≤ 4 → proper coloring ~~fo~~ of K vertices.



1, 3 have same colors

(2,4), (2,5) cannot be connected

Jordan's curve Theorem.

- Chromatic Partitioning

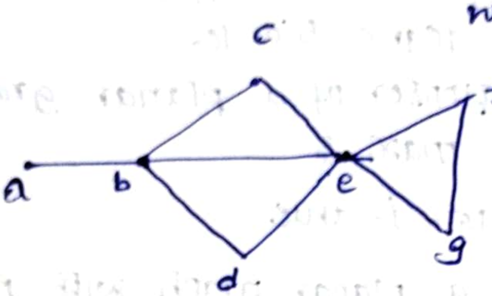
Distribute / partition the set of vertices of a simple connected graph into the minimum number of subsets s.t each subset of vertices can be coloured with the same color.

Independent set: A set of vertices s.t no two of them are adjacent → Trivial example any single vertex $u \in V$

Maximum Independent set :

An independent set where no more vertices can be added without violateding the independence property

Maximal IS with size 4



$\{a, c, d, f\}$ $\{b, g\}$

independent set → internally stable set

Largest minimal I.s cardinality = Independence Number = co-efficient of internal stability

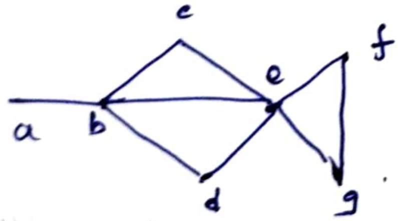
$\beta(G)$

K- chromatic graph of n vertices

$\beta(G) \geq \frac{n}{k}$

$\phi = \sum ab = 0$ ab : an edge (a)—(b)

$\overline{\phi} = 1 = \prod(\overline{a}+\overline{b}) = \underbrace{f_1 + f_2 + \cdots f_k}_{SOP} = 1$



~~f=~~ $\phi = ab + bc + bd + ce + be + de + ef + eg + fg = 0$

$\overline{\phi} = 1 = (\overline{a}+\overline{b})(\overline{b}+\overline{c})(\overline{b}+\overline{d})(\overline{c}+\overline{e})(\overline{b}+\overline{e})(\overline{d}+\overline{e})(\overline{e}+\overline{f})(\overline{e}+\overline{g})(\overline{f}+\overline{g})$

$\cancel{(\overline{a}\overline{b} + \overline{b} + \overline{b}\overline{c})}$

$$(\bar{a}\bar{b} + \bar{a}\bar{c} + \bar{b} + \bar{b}\bar{c})(\bar{b}\bar{c} + \bar{b}\bar{e} + \bar{d}\bar{c} + \bar{d}\bar{e})$$

$$(\bar{b}\bar{d} + \bar{b}\bar{e} + \bar{e}\bar{d} + \bar{e})(\bar{e} + \bar{e}\bar{g} + \bar{f}\bar{e} + \bar{f}\bar{g})(\bar{f} + \bar{g})$$

$$(\bar{b} + \bar{a}\bar{c})(\bar{e} + \bar{b}\bar{d})(\bar{e} + \bar{f}\bar{g})(\bar{f} + \bar{g})(\bar{b}\bar{c} + \bar{b}\bar{e} + \bar{c}\bar{e} + \bar{d}\bar{e})$$

$$(\bar{b}\bar{e} + \bar{b}\bar{d} + \bar{a}\bar{c}\bar{d} + \bar{a}\bar{b}\bar{c}\bar{d})(\bar{e}\bar{f} + \bar{e}\bar{g} + \bar{f}\bar{g} + \bar{f}\bar{g})$$

$$(\bar{b}\bar{c} + \bar{b}\bar{e} + \bar{c}\bar{e} + \bar{d}\bar{e})$$

$$\phi' = b'e'f' + b'e'g' + b'c'd'f'g' + a'c'e'f' + a'c'e'g'$$

Graph Problem

- Vertex Cover
- Clique
- TSP

} Approximation Algorithm

verfex cover → Min vertex cover $\gamma(G)$

↓

All the edges are incident to any of the vertices.

Independent set → No two them are adjacent (Thyey dont share any edge)

↳ $\beta(G)$

$n = \gamma(G) + \beta(G)$

Maximal Independent set

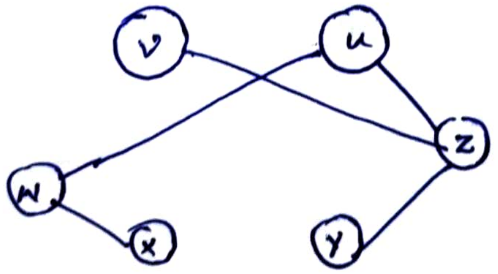
SAT [Boolean Algebra]

↓

complement form of vertices not present in Max I.S

$\{u, v, x, y\}$

G



vertex cover $\{w, z\}$

Algo Vertex Cover
{
  Algo of I.S
  {
  }
}

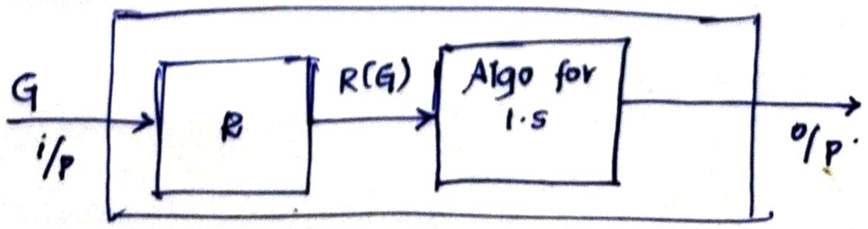
solution of I.S

⇓

solution of vertex cover

vertex cover $\overset{P}{\leq}$ I.S (exponential)

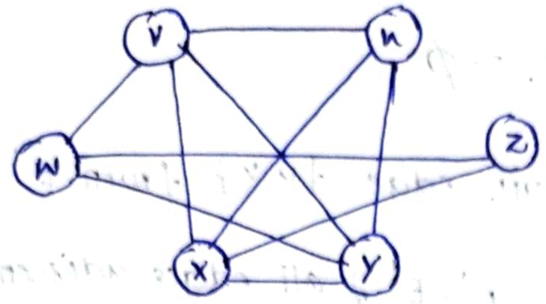
vertex Cover



Reduction of vertex cover to I.S

Clique → A set of vertices such that all the vertices are connected to each other.

G'



$\{u, v, x, y\}$.

$e \in E \rightarrow G$

$e' \in E' \rightarrow G'$ s.t $e' \notin E$

Vertex Cover of G

1. $G \rightarrow G'(V', E')$

$e' \in E'$ iff $e' \notin E \;\forall e'$

Graph where every possible edge except the edges of G are present

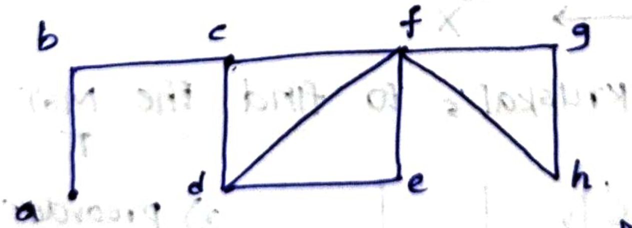
2. Find Clique of $G' \Rightarrow S'$

3. Take the complement set of vertices of G : $S = V - S'$

Vertex Cover - All edges adjacent to the chosen set of vertices

Matching

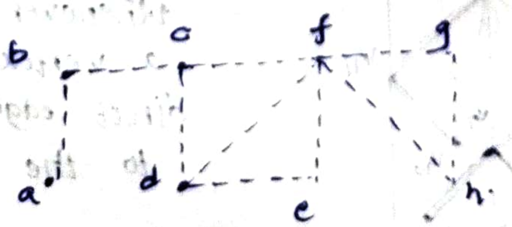
- set of edges s.t no two of them are adjacent



Matching set

Min vertex cover $\{b, d, f, g\}$

$M = \{\{ab\}, \{df\}, \{gh\}\}$



$C = \{a, b, d, f, g, h\}$

$M = \{\{ab\}, \{df\}, \{gh\}\}$ — Matching set

$C = \{a, b, d, f, g, h\}$

st 1 : $E' = E$, $M = \phi$, $C = \phi$

st 2 : While $E' \neq \phi$

chose a random edge $\{u, v\}$ from E'

$M = M \cup \{\{u, v\}\}$ $\quad E' = E' - \left\{ \begin{array}{l} \text{all edges adjacent} \\ \text{to } u \text{ or } v \end{array} \right\}$

$C = C \cup \{u, v\}$

Min vertex cover size cannot be lesser than the Maximal Matching set.

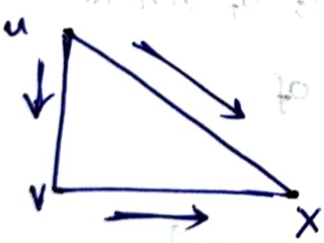
$$\text{Approximation Ratio} = \frac{\text{Actual sol}^n \text{ of Approx Algo}}{\text{optimal sol}} \leq \frac{2|M|}{|M|} \leq 2.$$

TSP

exponential time

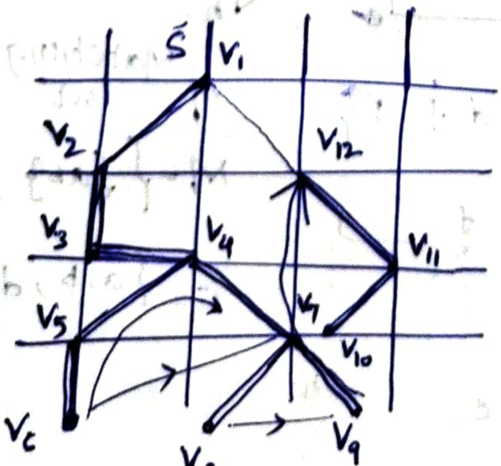
$\#$ possible solution $= \dfrac{(n-1)!}{2}$.

Approximate solution with Triangular inequality



$cost(u \xrightarrow[v]{} x) \geq cost(u \rightarrow x)$

1) Use prim's / kruskal's to find the MST for G
T



2) preorder Traversal of T following the edges of T as fas as possible, Whenever it is repeating a vertex, take the direct edge from one to the other

V_1 V_2 V_3 V_4 V_5, V_6, V_7, V_8, V_9, V_{12}, V_{11}, V_{10}

3) Take the last vertex of the traversal & form there direct edge to the source.

$$Ratio \leq \frac{cost(2 \times MST)}{MST} \leq 2$$