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Combinatorics and Graph Theory

unit - I : Basics of combinatorics.

Unit - II : Recurrence relation and generating functions

unit - III, IV, V : Graph theory

- Basics of Graph tree
- Euler / Hamiltonian Graph, plano and dual graph, Graph coloring
- NP hard, NP complete

NP - Non deterministic Polynomial

(Independent set, Dominating set, vertex cover, Approximation Algorithms)

Books:

Unit I, II - Rosen, Grimaldi

Unit - III, IV, V - Narasingh Deo, Cormen

(*) INDUCTION: ($\mathbb{Z}_+$)

→ Basis is true for the first +ve integer.

→ Induction hypothesis:

Inductive step: $\forall_K (P(K) \rightarrow P(K+1))$, $K \in \mathbb{Z}_+$

Assumption that for any arbitary integer K, if P(K) is true then P(K+1) is also true.

Basis condition and Induction hypothesis together comprises the Principal of Mathematical Induction.

P.T.O.

$$P(1)$$

$$\forall_{k \in \mathbb{Z}_+} \; P(k) \longrightarrow P(k+1)$$

$$\overline{\forall_n \; P(n), \; n \in \mathbb{Z}_+}$$

Secret sharing

Fall of dominos

Infinite ladder climbing

Eg-1: Sum of first n positive integers:

$$P(n): \; 1+2+\dots+n = \frac{n(n+1)}{2}$$

Basis: $P(1)$: $1 = \frac{1(1+1)}{2} = 1$ Basis is true.

Inductive step: $P(k)$: $1+2+\dots+k = \frac{k(k+1)}{2}$

$$P(k+1): \underbrace{1+2+\dots+k}_{\frac{k(k+1)}{2}} + k+1 = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2} + k+1 = (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$$

Eg-2: Sum of first n odd integers

$$1+3+5+\dots+(2n-1) = n^2$$

Basis: $P(1)$: $1 = 1^2$

Inductive step: $P(k)$: $1+3+5+\dots+(2k-1) = k^2$

$$P(k+1): \; 1+3+5+\dots+(2k-1)+(2k+1) = k^2+2k+1 = (k+1)^2$$

P.T.O

Eg - 3 : $P(n) : n < 2^n$

Basis: $P(1) : 1 < 2^1$

Inductive step : $P(k) : k < 2^k$

Add 1 to both sides : $k+1 < 2^k + 1$

We know $1 < 2^k$ $\therefore k+1 < 2^k + 1 < 2^k + 2^k$

$k+1 < 2^k + 1 < 2^{k+1}$

$\therefore k+1 < 2^{k+1}$ $P(k+1)$ is true.

Eg - 4 : Prove that $H_{2^n} \geq 1 + \frac{n}{2}$, n is a non-negative integer

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

Basis : $P(0) : H_{2^0} = H_1 = 1 + \frac{0}{2} = 1$

Inductive step : $P(k) : H_{2^k} \geq 1 + \frac{k}{2}$

$$H_{2^k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} \geq 1 + \frac{k}{2}$$

To prove : $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$

$$H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} + \underbrace{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^k+2^k}}$$

no. of terms $= 2^k$, least term $= \frac{1}{2^{k+1}}$

$$H_{2^{k+1}} \geq 1 + \frac{k}{2} + 2^k \times \frac{1}{2^{k+1}}$$

$$H_{2^{k+1}} \geq 1 + \frac{k}{2} + \frac{1}{2} \qquad \therefore H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$$

<u>Eg - 5</u> : Prove that for every set of lines in a plane where no two of which are parallel, meet at a common point.

(→ False statement)

Basis : $P(2)$ is true

Induction : $P(k)$ is true.

Now, for any set of k lines should meet at a point.

P_1 and P_2 cannot be distinct.

Here, by induction $P(k+1)$ is true and hence $P(n)$ is true for all n.

But $P(2) \not\Rightarrow P(3)$ INDUCTION FAILS.

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1) Prove that $1^2 + 3^2 + 5^2 + \dots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$ whenever n is a non-negative integer.

<u>sol</u>:- $1^2 + 2^2 + 3^2 + \dots + (2n+2)^2 = \cancel{\frac{n(n+1)(2n+1)}{6}}$

$$= \frac{(2n+2)(2n+3)(4n+5)}{6}$$

$$2^2 + 4^2 + 6^2 + \dots + (2n+2)^2$$

$$= 2^2\left(1^2 + 2^2 + 3^2 + \dots + (n+1)^2\right)$$

$$= 2^2 \frac{\left((n+1)(n+2)(2n+3)\right)}{6} = \frac{2}{3}(n+1)(n+2)(2n+3)$$

$$\therefore \frac{1}{3}(n+1)(2n+3)(4n+5) = 1^2 + 2^2 + \dots + (2n+2)^2$$

$$\therefore \frac{1}{3}(n+1)(2n+3)\left[(4n+5) - 2(n+2)\right]$$

$$= \frac{1}{3}(n+1)(2n+3)\left[4n+5-2n-4\right]$$

$$= \frac{1}{3}(n+1)(2n+3)(2n+1) = \frac{1}{3}(n+1)(2n+1)(2n+3)$$

Induction proof:

$n=0$, $\frac{1}{3}(1)(1)(3) = 1$ //

$1^2 = 1$. // True for $n=0$.

Now, try for $k+1$ assuming,

$$1^2 + 3^2 + 5^2 + \dots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$\underbrace{1^2 + 3^2 + 5^2 + \dots + (2k+1)^2}_{\frac{(k+1)(2k+1)(2k+3)}{3}} + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$$

$$\frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$$

$$(k+1)(2k+1) + 3(2k+3) = (k+2)(2k+5)$$

$P(k+1)$ is true

$$2k^2 + 3k + 1 + 6k + 9 = 2k^2 + 9k + 10$$

$$2k^2 + 9k + 10 = 2k^2 + 9 + 10$$

Hence Proved.

2) Prove that $3 + 3.5 + 3.5^2 + \cdots + 3.5^n = \frac{3(5^{n+1}-1)}{4}$ whenever n is a non-negative integer.

Sol:- Induction:

$P(0) \Rightarrow 3.1 = \frac{3}{4}(5-1) \Rightarrow 3 = 3$

Hence $P(0)$ is true.

$P(k) \Rightarrow 3 + 3.5 + 3.5^2 + \cdots + 3.5^k = \frac{3}{4}(5^{k+1}-1)$

$P(k+1) \Rightarrow 3 + 3.5 + 3.5^2 + \cdots + 3.5^k + 3.5^{k+1} = \frac{3}{4}(5^{k+2}-1)$

Assuming $P(k)$ is true,

$$\frac{3}{4}(5^{k+1}-1) + 3.5^{k+1} = \frac{3}{4}(5^{k+2}-1)$$

$$5^{k+1} - 1 + 4.5^{k+1} = 5^{k+2} - 1$$

$$5^{k+1} + 4.5^{k+1} = 5^{k+2}$$

$$1 + 4 = 5$$

$$5 = 5$$

$P(k+1)$ is true.

Hence proved.

It is true for all $n \in (Z^+ + \{0\})$

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Strong Induction :- (Second principle of mathematical induction)

* Base $P(1)$ must be true.
* $P(2), P(3), P(4) \ldots, P(k)$ is true. (each and every step is true).
* $$\frac{\left[P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k) \rightarrow P(k+1) \right]}{P(n) \ \forall \ n.}$$

Q. For any $n > 1$, n can be written as the product of prime.

Sol:- Base : $P(2)$ is true. → prime itself.

Inductive step : $P(j)$ is true for all $j \in \mathbb{Z}_+$ → all numbers upto j can be expressed as product of primes.

$P(j+1)$ → case 1 : prime number
$P(j+1)$ → case 2 : composite number.

Take case 2 : $j+1 = a * b$ (Composite)

$1 < a, b < j+1$

Therefore, either a or/and b are primes or they can be expressed as product of primes as per our inductive step.

Hence proved by Strong induction

Axioms for the set of ⊕ve integers:-

1) The number 1 is a positive integer.
2) If n is a ⊕ve integer, the successor of n, $n+1$ is also a ⊕ve integer.
3) Every ⊕ve integer other than 1 is the successor of a ⊕ve integer.
4) Every non empty subset of the set of ⊕ve integer has a least element (Well ordering property).

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If a is an integer and d is a ⊕ve integer, then $\exists$ integers (unique) q, r with $0 \leq r < d$ and

$$\boxed{a = dq + r.}$$

$$S = \{ a - dq \text{ for any } d, q \}$$

Ex 1 : Basis : $3 \in S$

Recursive step if $x, y \in S \Rightarrow x + y \in S$

Ex 2 : set of all letters over english alphabet

$$\Sigma_{\text{alphabet}} = \{ a, b, c, \ldots, g \}$$

Capital SIGMA

Set of all strings over the englix alphabet or any set. (Infinite set) $\Rightarrow$ Σ^* Kleene star

1) $\lambda \in \Sigma^*$ (empty string)

2) If $w \in \Sigma^*$, $x \in \Sigma$ then $wx \in \Sigma^*$

$\Sigma = \{a, b\}$ $\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aab, \ldots\ldots\}$

* Concatenation:

Ex - 3 :

1) If $w \in \Sigma^*$ then $w.\lambda = w$

2) If $w_1, w_2 \in \Sigma^*$, $x \in \Sigma \Rightarrow w_1.(w_2.x) = (w_1.w_2).x$

Ex - 4:

Length of a string:

1) $l(\lambda) = 0$ 2) $l(wx) = l(w) + 1$ where $w \in \Sigma^*$, $x \in \Sigma$.

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* Well formed formula (wff) :-

1) T, F, x (propositional variable) is a ~~cal~~ wff.

2) if x, y are wffs, then $(\neg x)$, $(\neg y)$, $(x \wedge y)$, $(x \vee y)$, $(x \rightarrow y)$ are also wffs.

3) Nothing else is an wff.

SIGMA

Task: p, q are propositional variable

$(pq \wedge)$ $(p \neg \wedge q)$, $(\neg \vee pq)$ are not wffs

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* <u>Structural induction:-</u>

Two sets are equal if $A \subseteq S$ and $S \subseteq A$.
$(A \& S)$

This can be used to prove that ~~all~~ ~~a~~ two sets are equal.

Let S: $3 \in S$ $\quad x, y \in S \longrightarrow x+y \in S = \{3, 6, 9, \ldots\}$
A: $\{3n, n \in N\}$

$A \subseteq S$

Basis: $n = 1 \Rightarrow 3 \in A$ and $3 \in S$

Inductive step: $x, y \in A$ $\Rightarrow$ $x = 3k_1$ $\quad x+y = 3(k_1 + k_2)$
$x, y \in S$ $\quad y = 3k_2$

$x+y = 3(k_1 + k_2) \in A$. Also, it belongs to S as it can be obtained by adding 3 and $3(k_1 + k_2 - 1)$ which is an element that can be obtained by $3 + 3(k_1 + k_2 - 1) + \ldots$ till $k_1 + k_2 - n = 1$ ~~where~~ at which point, $3(1) = 3$ ~~also~~ belongs to the set S. Hence $A \subseteq S$.

Similarly, for $S \subseteq A$ The elements of S are produced by $\{3\}$ as base.

The only possible elements are $\{3, 3+3, 3+3+3, \ldots\}$

Hence all of them can be expressed as multiples of 3.

$\therefore S \subseteq A$.

Therefore the sets $S = A$.

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* **ROOTED TREE :-**

Basis: If r is a single node, it is a rooted tree.

Recursive step: If $T_1, T_2, \ldots, T_k$ are some rooted trees with roots $r_1, r_2, \ldots, r_k$ then the new graph with a new root r having connection to the existing roots $r_1, r_2, \ldots, r_k$ is a rooted tree.

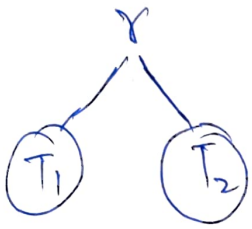
Complete binary tree: The tree must be complete upto $n-1$ level.

Full binary tree :- Every node has either 0 or 2 children.

Full binary tree:-

Basis : Single vertex

Recursive : T_1, T_2 are full binary trees, connect T_1 & T_2 as the children of a new root r to get a new full binary tree.



Height : $h(T)$

Basis : $h(T) = 0$ for single vertex.

Recursive : $h(T) = 1 + \max(h(T_1), h(T_2))$

Nodes : $n(T)$

Basis : $n(T) = 1$ for single vertex

Recursive : $n(T) = 1 + n(T_1) + n(T_2)$

Theorem : $n(T) \leq 2^{h(T)+1} - 1$ //

Basis : $n(T) = 1$, $h(T) = 0$ $\quad n(T) = 1 \leq 2^{0+1} - 1$

Inductive step : $n(T_1) \leq 2^{h(T_1)+1} - 1$

$n(T_2) \leq 2^{h(T_2)+1} - 1$

$n(T) = n(T_1) + n(T_2) + 1$ $\quad \therefore n(T) \leq 1 + 2^{h(T_1)+1} - 1 + 2^{h(T_2)+1} - 1$

$n(T) \leq 2^{h(T_1)+1} + 2^{h(T_2)+1} - 1$

$$n(T) \le \left(2^{h(T_1)+1} + 2^{h(T_2)+1}\right) - 1$$

$$n(T) \le 2 \cdot \left(\max\left(2^{h(T_1)+1}, 2^{h(T_2)+1}\right)\right) - 1$$

$$n(T) \le 2 \cdot \left(2^{h(T)}\right) - 1$$

$$n(T) \le 2^{h(T)+1} - 1 \quad //$$

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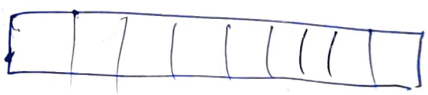
Counting

1) Product rule
2) Sum rule.



k length bitstring

Product rule.
2^k ways /

* No. of functions from Domain (m) to Codomain (n) is n^m.

* No. of one-one functions from domain to codomain is ${}^nP_m$.

* no. of elements in union set (sum rule).

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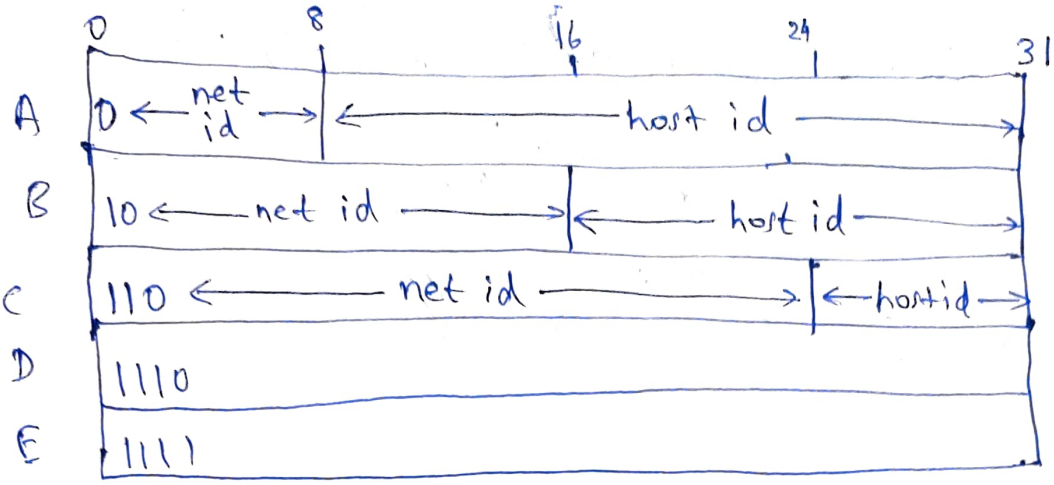
* IPv4 :-



0 1 1 1 1 1 1 1 × for net id.

0 . . . . . 0 } × for host id.
1 1

∴ A : $(2^7 - 1)(2^{24} - 2)$

B : $2^{14} \times (2^{16} - 2)$

C : $2^{21} \times (2^8 - 2)$

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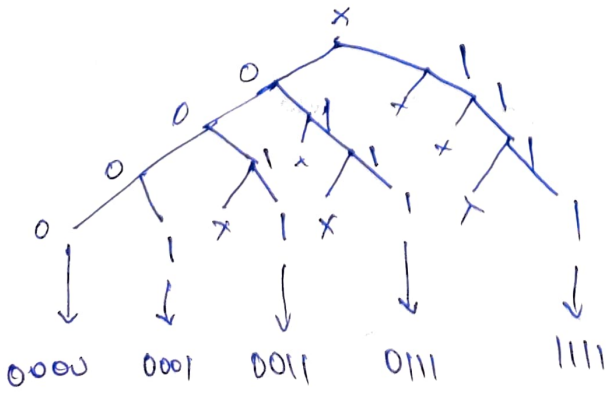
* Tree diagram :-



Combination
1 followed by 0
is invalid.

Example

* Principle of Exclusion and Inclusion (or) Subtraction principle of counting :-

Ex: 8 bit string

1 _ _ _ _ _ _ _ 2^7 Possibility-1

1 _ _ _ _ _ 0 0 2^5 Not allowed.

1 _ _ _ _ _ 0 0 2^6 Possibility - 2

Total possibilities $= 2^7 + 2^6 - 2^5$ //

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* Pigeonhole principle :-

$\underbrace{\square\ \square\ \square\ \ldots\ldots\ \square}_{K}$ $K+1$ objects

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For every integer n, there is a multiple of n that has only 0's and 1's in its decimal expansion form.

Take $n+1$ numbers $1, 11, 111, \ldots\ldots, 111\ldots\ldots1$

Divide by n $\Rightarrow$ Remainder $\underbrace{0 \text{ to } n-1}_{n \text{ numbers.}}$

Using pigeonhole principle, there must at least be 2 numbers x and y having the same remainder r.

$\therefore x = k_1 n + r$ $\quad x - y = n(k_1 - k_2)$

$y = k_2 n + r$ $\quad$ Hence proved.

$$\begin{array}{r} x \quad 111\ldots\ldots1 \\ y \quad 11\ldots\ldots1 \\ \hline 100\ldots\ldots0 \end{array}$$

08/08/24

(*) If N objects are placed into k boxes, there is atleast one box containing atleast $\left\lceil \frac{N}{k} \right\rceil$ objects. $N > k$.

Proof: $\left(\left\lceil \frac{N}{k} \right\rceil - 1 \right) k < \left(\left(\frac{N}{k} + 1 \right) - 1 \right) k = N$

* Show that among any $n+1$ (+)ve integers $\leq 2n$, there must be an integer that divides one of the other integer.

$\{ 2^{k_1} q_1, 2^{k_2} q_2, 2^{k_3} q_3, \ldots, 2^{k_{n+1}} q_{n+1} \}$

$2^{k_i} \times q_i \leftarrow q_i$ is odd $\Rightarrow$ n such odd numbers $\leq 2n$

$x = 2^{k_i} q_i$ assuming $k_i > k_j$ and $q_i = q_j = q$.

$y = 2^{k_j} q_j$

$\therefore \frac{x}{y} = 2^{k_i - k_j}$ //

(*) Theory: Every sequence of n^2+1 distinct real number contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Q. Assume there is a group of 6 people where every pair of them are either friends or enemies. Show that there are three mutual friends or three mutual enemies in the group.

Sol:- $\left\lceil \frac{6-1}{2} \right\rceil = 3$ ∴ By Pigeonhole principle, there is either a set of mutual friends or mutual enemies.

(RAMSEY NUMBER)

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* PERMUTATION AND COMBINATION:-

r - permutation
(ordered arrangement of the objects)

$$^nP_r = \frac{n!}{(n-r)!}$$

r - combination
(unordered arrangement)

$$^nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Binomial expression:

$$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \dots + {}^nC_n\, x^0 y^n$$

P.T.O.

* **Theorem:** $^{n+1}C_k = {}^{n}C_{k-1} + {}^{n}C_k$

$$\frac{(n+1)!}{(n+1-k)!\,k!} = \frac{n!}{(k-1)!\,(n+1-k)!} + \frac{n!}{k!\,(n-k)!}$$

$$\frac{n+1}{(n-k+1)} \times \frac{1}{k} = \frac{1}{n-k+1} + \frac{1}{k}$$

$$\frac{n+1}{k} = 1 + \frac{n-k+1}{k}$$

$n+1 = k + n-k+1$ $\qquad$ $n+1 = n+1$

LHS = RHS

Hence Proved.

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① $\sum_{k=0}^{n} {}^{n}C_k = 2^n$

② $\sum_{k=0}^{n} (-1)^k \left({}^{n}C_k\right) = 0$

③ $\sum_{k=0}^{n} 2^k \left({}^{n}C_k\right) = 3^n$

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④ $^{m+n}C_r = \sum_{k=0}^{r} {}^{m}C_{r-k} \times {}^{n}C_k \qquad m, n \geq r$

⑤ $^{2n}C_n = \sum_{k=0}^{n} \left({}^{n}C_k\right)^2$

⑥ $^{n+1}C_{r+1} = \sum_{i=r}^{n} {}^{i}C_r$

* Generated Permutation & combination :-

① r - permutation of n objects with repetition $= n^r$

② r - combination " " " " " $= {}^{n+r-1}C_r$

③ Permutation with identical object $= \dfrac{n!}{n_1!\, n_2! \cdots\cdots n_k!}$

(n_i objects of type i)

Distinguishable objects into distinguishable boxes

④ n indistinguishable objects into k indistinguishable boxes.

$a_1 + a_2 + \cdots + a_k = n \qquad a_1 \le a_2 \le \cdots \le a_k$

$\{a_1, a_2, \cdots, a_k\}$ Partition of $n = P_k(n)$

⑤ Distinguishable objects into indistinguishable boxes

(n) $\boxed{\text{STIRLING NO. OF SECOND KIND}}$ (k)

Q. 4 students 3 jobs. One student max. one job

${}^4C_3$ selecting students. ${}^4C_3 \times 3!$ ways.

$\therefore \dfrac{4!}{3!\,1!} \times 3! = 4! = 24$ (maybe wrong)

Solve using $S(4,3) = 6$.

Nigger

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Solving recurrence

$\{a_n\} = a_0, a_1, \ldots\ldots, a_{n-1}$

* **Towers of Hanoi:**

$$H_n = 2H_{n-1} + 1 \qquad H_0 = 0 \qquad n \geq 1$$

$$= 2^n - 1$$

* **Minimise the no. of multiplications needed.**

$(A_1 A_2 \ldots A_k)(A_{k+1} \ldots\ldots A_n)$

$$C_n = \sum_{i=0}^{n-1} (C_i \times C_{n-i})$$

$$C_n = \frac{{}^{2n}C_n}{n+1}$$

C_n is called Catalan number.

* Linear homogenous recurrence relation of degree k with constant coefficients:-

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k a_{n-k}$$

$$= \sum_{i=1}^{k} c_i \cdot a_{n-i}$$

P.T.O

Let $a_n = r^n$ be a solution.

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

$$\boxed{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0}$$

Characteristics Equation.

case-1 : $k=2$.

Case 1a :

$r^2 - c_1 r - c_2 = 0$ $\quad r_1$ and r_2 are the roots

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

Case 1b : $r_1 = r_2 = r$

$$a_n = r^n(\alpha_1 + n\alpha_2)$$

Ex-1: $a_n = a_{n-1} + 2a_{n-2}$

$r^2 - r - 2 = 0$ $\quad r^2 - 2r + r - 2 = 0$ $\quad r(r-2) + 1(r-2)$

$(r+1)(r-2) = 0$ $\quad r = -1, r = 2.$

$$a_n = \alpha_1(-1)^n + \alpha_2(2)^n$$

P.T.O

Nigger

Q. $a_n = 6a_{n-1} - 9a_{n-2}$ $\quad a_0 = 1,\ a_1 = 6.$

Sol:

$$r^2 - 6r + 9 = 0$$
$$r^2 - 3r - 3r + 9 = 0$$
$$r(r-3) - 3(r-3) = 0$$
$$(r-3)(r-3) = 0 \qquad r = 3.$$

$$a_n = (\alpha_1 + \alpha_2 n)\, 3^n \qquad a_0 = 1$$

$$a_1 = (1 + \alpha_2)\, 3 \qquad a_0 = \alpha_1 = 1$$
$$a_1 = 6$$

$$2 = 1 + \alpha_2$$
$$\alpha_2 = 1 \qquad \therefore\ a_n = (n+1)\, 3^n$$

Case - 2 : $K = K$ (general case)

$r_1, r_2, \ldots r_K$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n + \ldots + \alpha_K r_K^n$$

Assume r_1 repeats m_1 times and till r_K repeats m_K times.

$$a_n = \left(\alpha_{1,0} + n\alpha_{1,1} + n^2\alpha_{1,2} + \ldots + n^{m_1-1}\alpha_{1,m_1-1}\right) r_1 +$$
$$\left(\alpha_{2,0} + n\alpha_{2,1} + \ldots + n^{m_2-1}\alpha_{2,m_2-1}\right) r_2 +$$
$$\vdots$$
$$\left(\alpha_{K,0} + n\alpha_{K,1} + \ldots + n^{m_K-1}\alpha_{m_K-1}\right) r^K$$

21/08/24

Q. $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ $\quad a_0 = 2$, $a_1 = 5$, $a_2 = 15$

Sol:- $r^3 - 6r^2 + 11r - 6 = 0$

$(r-1)(r-2)(r-3) = 0 \qquad r = 1, 2, 3$

$a_n = \alpha_1 + \alpha_2 (2)^n + \alpha_3 (3)^n$

$\alpha_1 + \alpha_2 + \alpha_3 = 2 \qquad \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$

$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \qquad \alpha_2 + 2\alpha_3 = 3$

$2\alpha_2 + 6\alpha_3 = 10 \qquad \alpha_2 + 3\alpha_3 = 5$

$\boxed{\alpha_3 = 2}$

$\boxed{\alpha_2 = -1}$

$\boxed{\alpha_1 = 1}$

$a_n = 1 - 2^n + 2 \cdot 3^n$

Q. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

$r^3 + 3r^2 + 3r + 1 = 0 \qquad (r+1)^3 = 0 \qquad r = -1, -1, -1$

$a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3)(-1)^n$ $\quad a_0 = 1$, $a_1 = -2$, $a_2 = -1$

$\boxed{1 = \alpha_1} \qquad a_n = (1 + n\alpha_2 + n^2\alpha_3)(-1)^n$

$-2 = (1 + \alpha_2 + \alpha_3)(-1) \qquad \alpha_2 + \alpha_3 = 1$

$-1 = 1 + 2\alpha_2 + 4\alpha_3 \qquad -\alpha_2 - 2\alpha_3 = 1$ (+)

$\alpha_3 = 2$

$\boxed{\alpha_3 = -2}$

$\boxed{\alpha_2 = 3}$

$-1 = \alpha_2 + 2\alpha_3$

Q. $f_n = f_{n-1} + f_{n-2}$

sol:- $a_n = a_{n-1} + a_{n-2}$ $\quad r^2 - r - 1 = 0$

$\frac{1 \pm \sqrt{1+4}}{2} \Rightarrow \frac{1 \pm \sqrt{5}}{2}$

Linear non-homogenous recurrence relation with constant coefficients :-

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

$a_n = a_n^p + a_n^h$ $\quad a_n^h$ is the solution of the associated homogenous recurrence relation.

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$$

case-1: s is not a root of the linear homogenous recurrence relation.

$$a_n^p = (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$$

case-2: s is a root of the recurrence relation with multiplicity 'm'.

$$a_n^p = (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n \times n^m$$

Q. $a_n = 3a_{n-1} + 2n$

$a_n^h = \alpha \cdot 3^n$ $\quad F(n) = 2n = (2n+0)\,1^n$

$a_n^p = (P_1 n + P_0)\,1^n$

$\therefore P_1 n + P_0 = 3(P_1(n-1) + P_0) + 2n$

$P_1 n + P_0 = 3P_1 n - 3P_1 + 3P_0 + 2n$

$-2P_1 n = -3P_1 + 2P_0 + 2n$

$(-2P_1 - 2)n + (3P_1 - 2P_0) = 0$

$-2P_1 - 2 = 0$ $\quad 3P_1 - 2P_0 = 0$

$P_1 = -1$ $\quad -3 - 2P_0 = 0$ $\quad P_0 = -\frac{3}{2}$ //

$a_n^p = \left(-n - \frac{3}{2}\right)1^n$ $\quad a_n^h = \alpha \cdot 3^n$

$a_n = \alpha \cdot 3^n - \left(n + \frac{3}{2}\right)$ $\quad a_1 = 3$ (given)

$3 = \alpha \cdot 3 - \frac{5}{2}$ $\quad \frac{11}{2} = \alpha \cdot 3$ $\quad \alpha = \frac{11}{6}$ //

$$a_n = \frac{11}{2} \cdot 3^{n-1} - \left(n + \frac{3}{2}\right)$$

Q. $a_n = 5a_{n-1} + 6a_{n-2} + 7^n$

sol: $r^2 - 5r - 6 = 0$ $\quad (r-2)(r-3) = 0$

$r = 6, -1$

$a_n^h = \alpha_1 (6)^n + \alpha_2 (-1)^n$

$b_0 = 1$, $S = 7$

$a_n^p = P_0 (7)^n$

$P_0 (7)^n = 5[P_0 7^{n-1}] + 6[P_0 7^{n-2}] + 7^n$

$P_0 \times 49 = 5P_0 \times 7 + 6P_0 + 49$

$49P_0 = 41P_0 + 49$ $\quad 8P_0 = 49$ $\quad P_0 = \frac{49}{8}$

$a_n^p = \frac{7^{n+2}}{8}$ $\quad a_n = \alpha_1 (6)^n + \alpha_2 (-1)^n + \frac{1}{8} \cdot 7^{n+2}$

---

* Master's Theorem :-

$T(n) = a\,T\left(\frac{n}{b}\right) + F(n)$

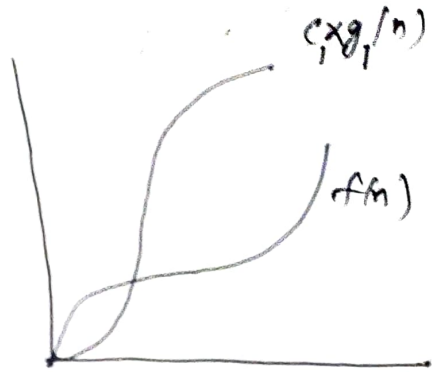
T = Time to search an element in an array.

* Asymptotic Notation :-

UPPER BOUND O ⇒ Big Oh

LOWER BOUND Ω ⇒ Big Omega

θ ⇒ Big Theta



$$f(n) \le c_1 \times g_1(n) \quad \forall \; n > n_0$$

$$f(n) = O(g(n))$$

$$f(n) = a_n = 5n^2 + 10n + 50$$

$g_1(n) = n^2$ when $f(n) \le c \cdot g_1(n)$

$$f(n) \ge c_2 \times g_2(n) \quad \forall \; n > n_0'$$

$$f(n) = \Omega(g_2(n))$$

If $c_1 g(n) \le f(n) \le c_2 g_2(n)$

$$f(n) = \theta(g(n))$$

---

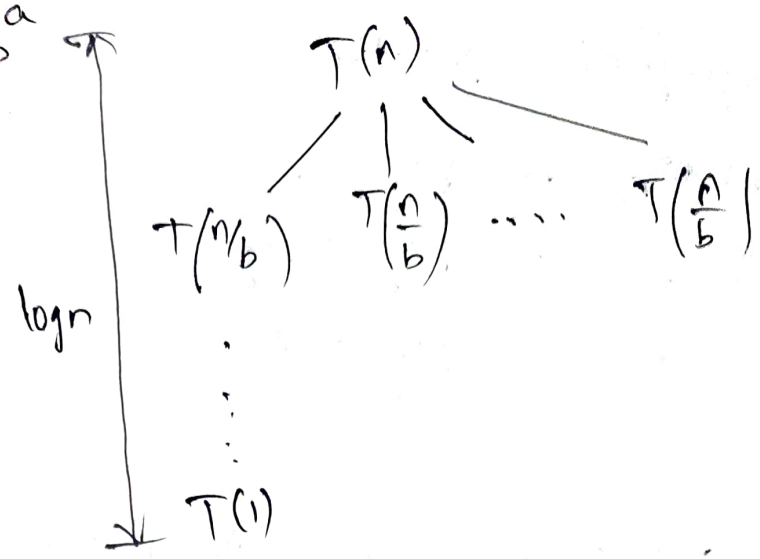
1) If $F(n) = O(n^{(\log_b a) - \epsilon})$ for some constant $\epsilon > 0$

then $T(n) = \theta(n^{\log_b a})$

2) If $F(n) = \theta(n^{\log_b a})$ then $T(n) = \theta(n^{\log_b a} \log n)$

(Continuation of Master's theorem)

$F(n) \sim n^{\log_b a}$



3) If $F(n) = \Omega\left(n^{(\log_b a) + \epsilon}\right)$ for some $\epsilon > 0$

and if $a \cdot F\left(\frac{n}{b}\right) \leq c \cdot F(n)$

for some $c > 1$ then $T(n) = \Theta(F(n))$.

Q. $T(n) = 9T(n/3) + n$

Sol:- $F(n) = n$ $\quad a = 9 \quad b = 3$

$T(n) = \Theta\left(n^{\log_b a}\right) = \Theta(n^2)$.

Q. $T(n) = T\left(\frac{2n}{3}\right) + 1$

Sol:- $F(n) = 1$ $\quad a = 1, \quad b = 3/2$

$n^{\log_{3/2} 1} = n^0 = 1$ $\quad n^{\log_b a} = F(n)$

$\therefore$ apply rule (2),

$$T(n) = \Theta\left(n^{\log_b a} \cdot \log n\right)$$

$$= \Theta\left(F(n) \cdot \log n\right)$$

$$= \Theta\left(1 \cdot \log n\right)$$

$$= \Theta\left(\log n\right)$$

Q. $T(n) = 3T\left(\frac{n}{4}\right) + n\log n$

Sol:- $F(n) = n\log n$ $\quad a = 3 \quad b = 4$

$n^{\log_b a} = n^{\log_4 3}$ $\quad n\log n > n^{0.9}$

Q. $T(n) = \sqrt{2}\,T(n/2) + \log n.$

Sol:- $a = \sqrt{2} \quad b = 2$ $\quad n^{\log_b a} = n^{\log_2 \sqrt{2}} = n^{1/2}$

Compare with $F(n)$. $\quad \log n < n^{1/2}$

$\therefore T(n) = \Theta\left(n^{1/2}\right)$

Q. $T(n) = 2T(\sqrt{n}) + 1$ if $n > 2$ else $T(n) = 2$

Sol:- $n = 2^k$ Assumption $T(2^k) = S(k)$

$\therefore S(k) = 2S(k/2) + 1$ $\quad F(n) = 1$

$a = 2, \quad b = 2$ $\quad k^{\log_2 2} = k$ $\quad = \boxed{\Theta(\log n)} =: \Theta(S(k))$

Q. $T(2^k) = 3T(2^{k-1}) + 1$

Sol:- $T(2^k) = 3T(2^k/2) + 1$

$a = 3 \quad b = 2 \qquad (2^k)^{\log_2 3} > 1$

$\therefore \Theta((2^k)^{\log_2 3}) = \Theta(2^{\log_2 3^k}) = \boxed{\Theta(3^k)}$

Q. $T(n) = 2T(n/2) + n \log n$.

Sol:- $a = 2, b = 2 \qquad n^{\log_2 2} = n^1$

$n < n \log n \qquad F(n) = n \log n$

$n \log n \leq c \cdot n \log n$

$c \geq 1 \qquad$ ~~$f(n) = \Theta(n \log n)$~~

Since it is not strictly greater than, cannot be solved apparently.

Q. $f(n) = 5 \cdot f(n/2) + 3$

Sol:- $a = 5, b = 2, \quad F(n) = 3$

$n^{\log_2 5} > 3 \qquad \therefore \Theta(n^{\log_2 5})$

Q. $f(n) = 2f(n/2) + 2$

Sol:- $a = 2, b = 2 \qquad n^{\log_2 2} = n^1 \quad F(n) = 2$

$n > 2 \quad \therefore \Theta(n)$

Q. $f(n) = 2f(n/2) + n$ (merge sort)

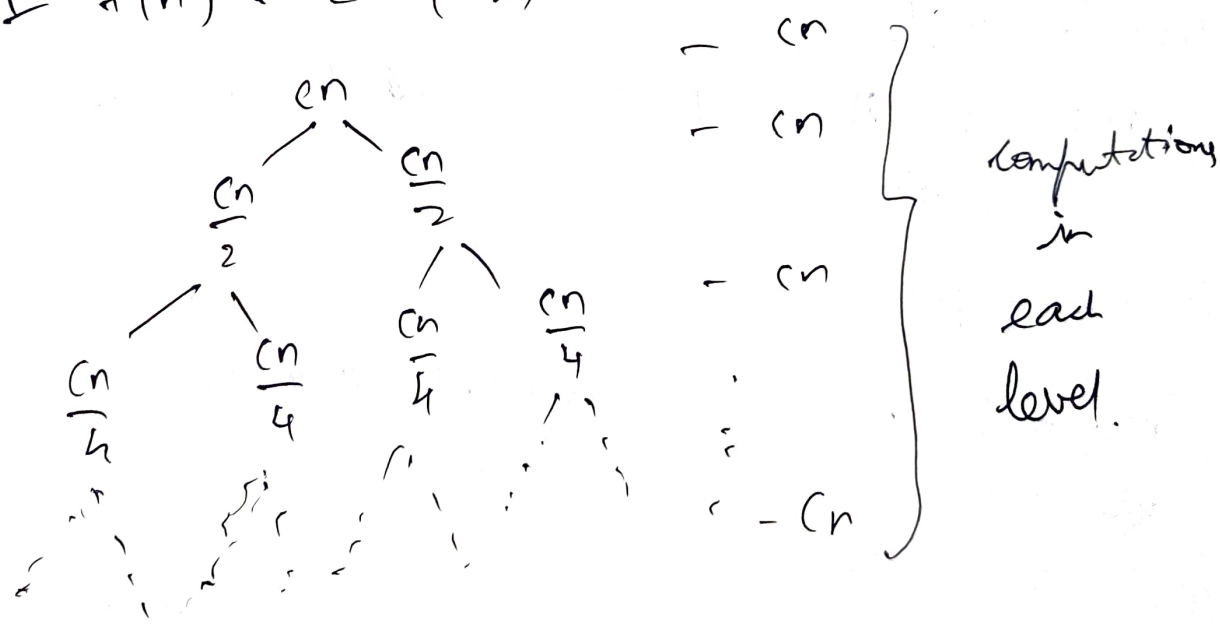
$n = n$ $\therefore \Theta(n \times \log n)$

Q. $f(n) = 7f(n/2) + \frac{15}{9}n^2$ (Strassen's formula)

Sol:- $n^{\log_2 7}$ $\frac{15}{9}n^2$

$\approx n^3 > \frac{15}{9}n^2$ $\therefore \Theta(n^{\log_2 7})$ //

Q. $f(n) = 2f(n/2) + cn$



No. of levels $= \log_2 (cn)$ $\therefore$

$\therefore$ Total computation time $= cn \times \log_2(cn)$

$= cn \times (\log_2 n + \log_2 c)$

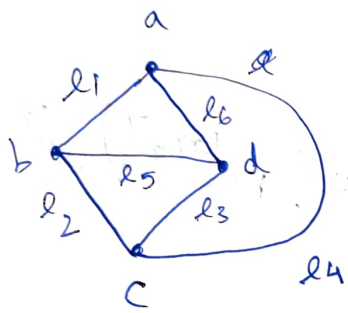
remove constant terms $= \Theta(n \log n)$ //

04/09/23

Graph Theory

* Graphs are not undirected unless the set is symmetric

* Graph theory is the study of sets.



V no. of vertices

e no. of edges

Incidence matrix:

$$M(G) = [m_{ij}]_{v \times e}$$

Undirected graph:

$m_{ij} = 1$ if edge e_j is incident on v_i

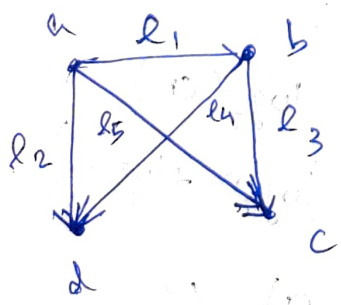
0 otherwise.

Directed graph:

$m_{ij} = 1$ if e_j is incident out of v_i

$= -1$ if e_j is incident to v_i

0 otherwise.



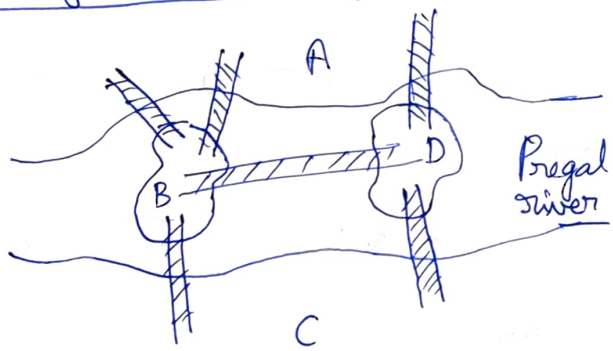
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

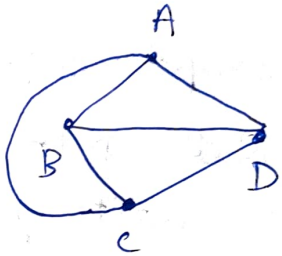
* **Adjacency matrix :-**

$$A(G) = [a_{ij}]_{V \times V} \qquad a_{ij} = 1 \text{ if } V_i \rightarrow V_j$$
$$0 \text{ otherwise.}$$

* Hamiltonian circuit, Hamiltonian path, Euler circuit, Euler path.

* **Königsberg Bridge Problem :-**





Theorem : A connected graph is an Euler graph (having an Euler circuit) iff all the vertices are of even degree.

Corollary : If a graph has exactly two vertices of odd degree it is having Euler path.

Th-1 : No. of vertices of odd degree in a graph is even.

Th-2 : A simple graph with n vertices and K components have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Th-3 : If there are exactly 2 vertices having odd degree in a graph, there must be an edge connecting these vertices.

$$\underbrace{\sum d_{odd}}_{\text{must be even}} + \underbrace{\sum d_{even}}_{\text{even}} = \underbrace{2|E|}_{\text{even no.}}$$

N. Deo - Graph theory :-

* ~~Complete~~ Simple graph : No self loops or parallel edges.
* Regular graph : Add vertices have same degree
* Complete graph : $\forall v_i, v_j \in V$, $v_i \to v_j$ is present in G.
* Null graph : No edges
* Bipartite graph : $G = G_1 \cup G_2$ $\quad G_1 =$ Null graph, $G_2 =$ Null graph.

But $\forall v_i \in G_1$ & $\forall v_j \in G_2$

$v_i \to v_j \in G$.

Kuratowski's graph.

* Travelling Salesperson Problem :-

Hamiltonian cycle over an weighted graph with minimum cost.

Total number of Hamiltonian circuits in a complete graph with n vertices is given by :-

$$\frac{1}{2}(n-1)! \quad //$$

P.T.O

* Havel Hakimi Algorithm :-

→ Solves graph realization problem.

Given a seq ( ) be a finite list of non-negative integers which is non-increasing.

Let $A' = (t_1 - 1, t_2 - 1, \ldots, t_s - 1, d_1, d_2, \ldots, d_n)$ be another list of non-negative integers.

Ex: $(6, 3, 3, 3, 3, 2, 2, 2, 1, 1) = A$

Sol:- Remove the first element, let it be m. and reduce the value of the next m numbers by 1.

$\therefore (\underbrace{2, 2, 2, 2, 1, 1}, 2, 1, 1) = A_1$

Now, rearrange $\Rightarrow (2, 2, 2, 2, 2, 1, 1, 1, 1)$

Apply the same process $\Rightarrow A_2 = (1, 1, 2, 2, 1, 1, 1, 1)$

$\Rightarrow (2, 2, 1, 1, 1, 1, 1, 1)$

$\Rightarrow A_3 = (1, 0, 1, 1, 1, 1, 1) \Rightarrow (1, 1, 1, 1, 1, 1, 0)$

Now remove 1. $A_4 = (0, 1, 1, 1, 1, 0)$

$\Rightarrow (1, 1, 1, 1, 0, 0) \Rightarrow A_5 = (0, 1, 1, 0, 0) \Rightarrow (1, 1, 0, 0, 0)$

$A_6 = (0, 0, 0, 0)$ //

$A = (6, \underbrace{5, 5, 4, 3, 2, 1})$

Sol:- $A_1 = (4, \underbrace{4, 3, 2, 1}, 0)$

$A_2 = (3, \underbrace{2, 1, 0}, 0)$

$A_3 = (1, 0, -1, 0)$ ⇒ Not a graphing sequence

~~Fleury's Algorithm:~~

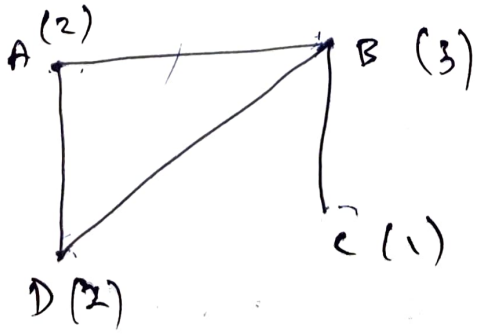
Fleury's Algorithm:-

1) If the graph is disconnected, exit with "Not an Euler graph".

2) If graph is having every vertex even degree or 2 vertices of odd degree, goto Step-3

3) Start with one of the two vertices of odd degree (or any vertex of even degree). For Euler circuit

Ex:

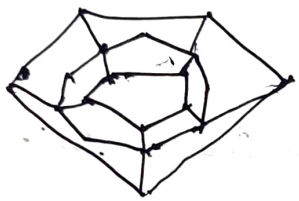


4) Traverse to a point and remove the corresponding edge from the graph.

~~2) If that step makes the graph~~

5) Repeat step 4,5 unless no edges are left.

6) Traversed edges form an Euler cycle if all the vertices are of even degree, otherwise it is an Euler path.



Dodecahedron.

11/09/2024

* <u>Shortest Path Algorithm</u> :-

— Dijkstra's Algorithm:

Step-1 : ① $P = \{0\}$. ② $T = V \setminus \{s\}$

③ $\forall v \in T$, $l(v) = \omega(s, v)$

Step-2 : select the vertex with min $l(v)$, let that vertex is 'a'.

Step-3 : If 'a' is destination, otherwise set

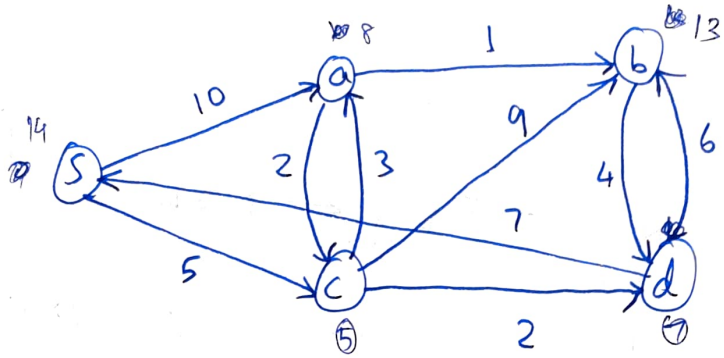
① $P' = P \cup \{a\}$ ② $T' = T \setminus \{a\}$.

③ $\forall v \in T$, $l'(v) = \min(l(v), l(a) + \omega(a, v))$

step-4: set $T = T'$ and $P = P'$

repeat step-2 and 3.

Ex:



Sol:- $P = \{s\}$ $T = \{a, b, c, d\}$

$l_{sa} = 10$ $l_{sc} = 5$

$\therefore$ $P = \{s, c\}$ $T = \{a, b, d\}$

$l_{ca} = 3$ $l_{cd} = 2$ $l_{cb} = 9$

$P = \{s, c, d\}$ we have reached destination

Hence, stop the algorithm.

Distance $= 7$ $\{s \to c + c \to d\}$.

---

* Shortest Path Problem :-

→ single source single destination shortest Path. (1)

→ single source All destination shortest path (2)

→ All source single destination shortest path. (3)

$O(V+E) \log V$ for (2)

* Bellman - Ford algorithm :- (Dynamic Programming)

→ Allows negative weight

→ Detects negative cycle

① BF (G, w, s) $\forall$ $v \in V[G]$ $d[s] \leftarrow 0$

$d[v] \leftarrow \infty$ $\pi[v] \leftarrow NIL$

distance = d s = source v = vertex.

G = graph w = weights.

$\pi(v)$ extra space utilised for vertex.

② For i = 1 do $\|V[G]\| - 1$

do for each $e = (u, v) \in E[G]$

if $d[v] > d[u] + w(v, u)$

$d[v] \leftarrow d[u] + w(v, u)$

$(\pi[v] \leftarrow u)$

③ for each edge $e = (u, v) \in E[G]$

if $d[u] > d[v] + w(u, v)$

* Bellman - Ford algorithm :- (Dynamic Programming)

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① BF (G, w, s) $\forall$ $v \in V[G]$ $\quad d[s] \leftarrow 0$

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distance = d $\quad$ s = source $\quad$ v = vertex.

G = graph $\quad$ w = weights.

$\pi(v)$ extra space utilised for vertex.

② for i = 1 to $\|V[G]\| - 1$

do for each $e = (u, v) \in E[G]$

if $d[v] > d[u] + w(v, u)$

$(\pi[v] \leftarrow u)$ $\quad d[v] \leftarrow d[u] + w(v, u)$

③ for each edge $e = (u, v) \in E[G]$

if $d[u] > d[v] + w(u, v)$

* Tree :-

→ Acyclic graph / Circuitless graph :-

→ n vertex tree → n-1 no. of edges.

→ Minimally connected graph

A tree with 2 or more vertices has atleast two pendant vertices.

* Spanning tree :-

→ skeleton of a graph.

→ Maximal tree subgraph.

* Rank (r) = no. of branches = $n - k$

& nullity (μ) = no. of chords = $e - n + k$

$= e - r$.

$n = |V|$ $\quad e = |E|$

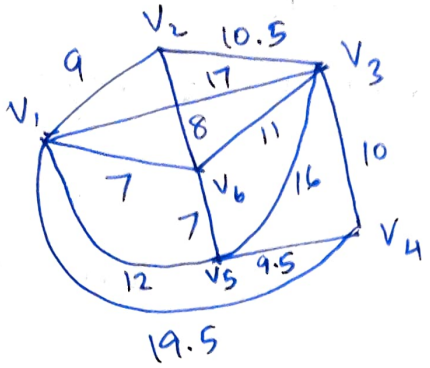
* Min. spanning tree :-

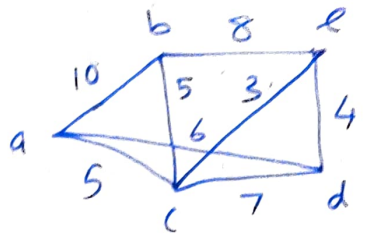
→ Kruskal's algorithm.

→ Prim's algorithm.

P.T.O

* Prim's algorithm :-





initially : ~~Rows~~ = starting point = {a}

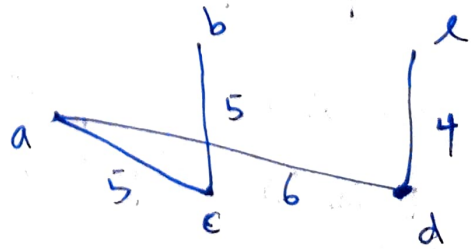
	a	b	c	d	e
a	–	10	5	6	∞
b	10	–	~~5~~	∞	8
c	~~5~~	5	–	7	7
d	~~6~~	∞	7	–	4
e	∞	8	7	4	–

i) Start with any row choose the minimum in the row. cross ~~ii) Take the rows with minimum out its compliment.~~

ii) Now update the rows to consider the chosen column as well.
Hence {a, c}.

iii) Now consider the rows corresponding to the values in "Rows".

1 → {a}
2 → {a, c}
3 → {a, c, b}



iv) ~~stop~~ when row. size = no of vertices.

* Cayley's theorem :-

Labelled trees : n^{n-2} (connected graph)

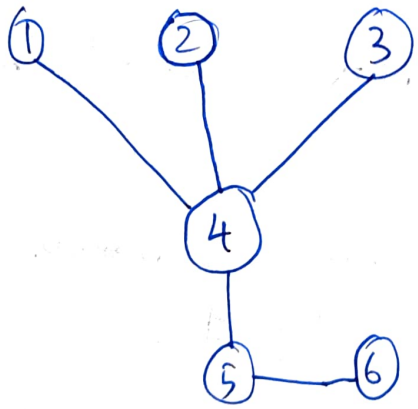
For a given n vertices no. of ways to create a tree.

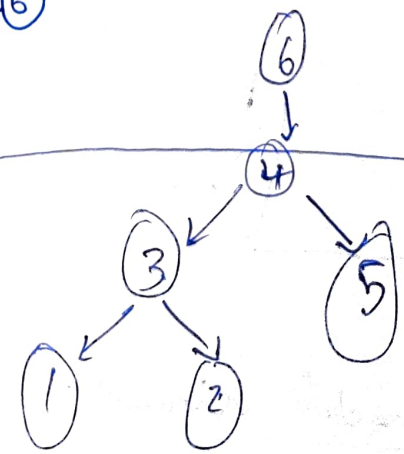
K - component graph : $\boxed{T_{n,k} = k \cdot n^{n-k-1}}$

Spanning tree for n vertex complete graph

$\Rightarrow n^{n-2}$

* Prufer's Encoding :-





PrC = {1, 1, 5}

{~~1~~, ~~1~~, ~~3~~, ~~4~~}

{~~2~~, 5, 6}

2 — 4 — 6

3 — 1 — 5

{6}

{4, 6}

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Cayley's ⟵ Spanning Tree ⟶ Prufer's encoding

No. of unique prufer sequence $= n^{n-2}$

$f : T_n \to P_n$

$f^{-1} : P_n \to T_n$

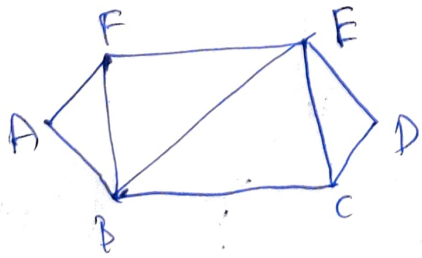
$f \circ f^{-1} = ID \qquad |T_n| = |P_n| = n^{n-2}$

Only for a complete graph no. of spanning tree is found.

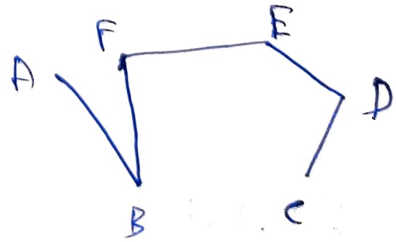
* Cutset & cut vertices :-

— Min no of edges whose removal makes the graph disconnected.

— Min no. of edges whose removal makes the rank of graph reduce by 1.



⇒



* Fundamental circuit & Fundamental cutset :-

Refer some notes.

Theorem: with respect to a spanning tree T, a chord c that determines a fundamental circuit T appears in fundamental ~~circuit~~ cutset associated with the branches of T and nowhere else.

Theorem: wrt T, a branch b that determines the fundamental cutset S, is contained in every fundamental circuit associated with T,

$T = \{b, f, g, d, i\}$

Fundamental circuits:

a: {

d: {a, c, d}

i: {i, h}

c: {c, f, g, h}

Fundamental cutsets:

b = {b, a}

f = {c, f, e, a}

Edge connectivity : Cardinality of min cutset of a connected graph.

Th: Edge connectivity cannot exceed the degree of the vertex with the least degree.

Vertex connectivity : Remove min degree of vertex to make the remaining graph as disconnected.

Separability : If vertex connectivity is 1,

but vertex : single vertex which makes the graph separable.

* Planar graph and its Duals :-

Embedding:

A drawing of a geometric representation of a graph on a surface, such that no two edges intersect.

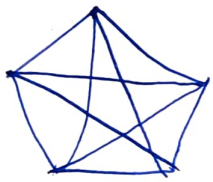
Planar graph:

X-Y plane embedding of a graph

Kuratowski ??

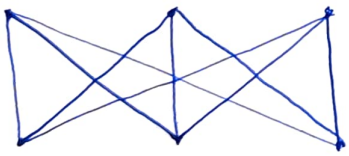
But,

$n = 5$
$e = 10$



K_5

cannot be converted into planar form.



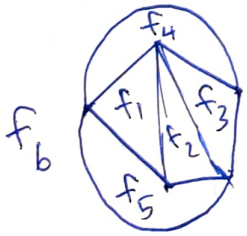
$K_{3,3}$

$K_{3,3}$ and K_5 are Kuratowski's graphs.

$n = 6$
$e = 9$

f denotes faces / region.

degree of a face = edges covering the face.



Euler's theorem:

$$\boxed{f = e - n + 2}$$

$n + f - e = 2$ → Euler's characteristic

Necessary conditions for a planar graph:-

$e \geq \frac{3}{2} f$ $\qquad$ $e \leq 3n - 6$

Bipartite graph:-

$e \leq 2n - 4$

Sum of degree of faces = sum of degree of vertices.

$\sum d(f_i) = 2e$

Th: A necessary and sufficient condition for a graph to be a planar graph is if doesnot contain either of K_5 or $K_{3,3}$ or any graph homeographic to either of them.

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* **Coloring:-** (chromatic number)

For a binary tree : 2.

Adjacent vertices cannot be given the same color.

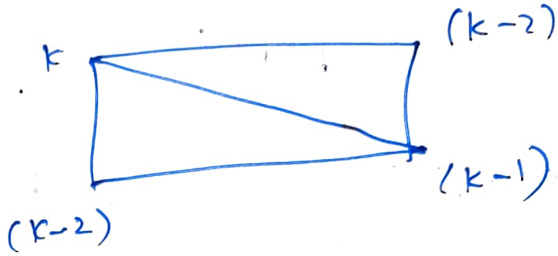
chromatic no. $\leq 1 + d_{max}$ $\qquad$ d_{max} = max (degree)

* If there is no complete subgraph of $d_{max} + 1$ vertices.

* Chromatic polynomial :-

No. of ways all elements of V can be colored using k - colors.

eg : $k(k-1)(k-2)^2$



$$P_{Graph}(k) = \underbrace{k(k-1)(k-2)\cdots(k-n+1)}$$

if it is of this form, the graph is a complete graph.

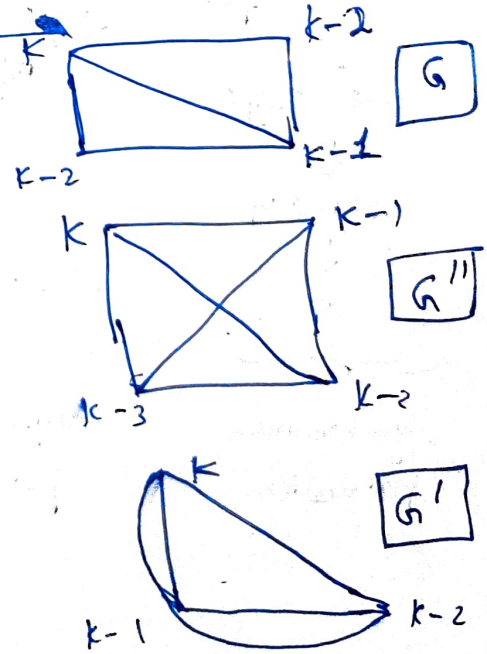
$$P_T(k) = k(k-1)^{n-1}$$ for a tree.

* Fundamental reduction theorem :-

G : given graph.

G' : same color

G'' : different color



one vertex removed. edges remain same.

converted to complete graph.

$$P_G(k) = P_{G'}(k) + P_{G''}(k)$$

$$k(k-1)(k-2)^2 = k(k-1)(k-2) + k(k-1)(k-2)(k-3)$$

$$k(k-1)(k-2)^2 = k(k-1)(k-2)\,[1+(k-3)]$$

$$= k(k-1)(k-2)^2$$

$$P_G(x) = a_p x^p + a_{p-1} x^{p-1} + a_{p-2} x^{p-2} + \dots$$

a) Degree of $P_G(x) = |V| = n$

b) $a_p = 1$ (monic polynomial)

c) Constant term ($a_0 = 0$)

d) Either $P_G(x) = x^n$ or sum of coefficients $= \sum_{i=0}^{p} a_i = 0$.

$x = 0$ No color $\Rightarrow P_G(x) = a_0 = 0$

$x = 1 \Rightarrow P_G(x) = \sum_{i=0}^{p} a_i$

Base: 1 edge, $n-2$ isolated vertices

$$P_G(x) = x(x-1)x^{n-2}$$

P.T.O

* Four color theorem :-

$$e \le 3n - 6$$

$$\frac{2e}{n} \le \frac{6n-12}{n}$$ (avg degree of a vertex.)

$$\text{Avg. deg of vertex} \le 6\left(1 - \frac{2}{n}\right) \quad \therefore \quad \underline{\underline{< 6}}$$

$\therefore$ Each vertex can have a maximum degree of 5, in a planar graph.

<u>Th</u>: The chromatic no. of a planar graph of n vertices is at max 5.
($\rightarrow$ computer based proof)

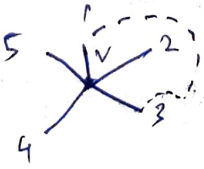
<u>Basis</u>: $n = 1$, $5 \Rightarrow$ Basis is true.

<u>Induction step</u>: Assume for a planar graph with $n = k-1$ vertices, 5 colors are sufficient for proper coloring [I.H].

Take a graph with k vertices, remove $v \rightarrow$ Remaining graph can be properly colored.

$\exists$ v which has degree 5 or less,

There can be many paths between edges but they end up



isolating an vertex from another, thereby allowing repetition of colors.

* Jordan's Curve theorem

* Chromatic ~~Partition~~ Partitioning :-

Distribute / Partition the set of vertices of a simple connected graph into the minimum no of subsets such that each subset of vertices can be colored with the same color.

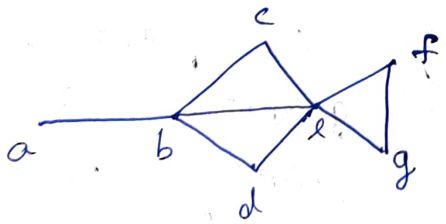
* Independent set :-

A set of vertices such that at no two of them are adjacent →

* Maximal I.S :- (Not unique).

An independent set where no more vertices can be added without violating the independence property.

Ex:



{a, c, d, f} - ①

{b, ~~g~~ f} - ②

{b, g} - ③

⋮

Minimal is ①,

Largest maximal I.S cardinality = Independence no

= coefficient of internal stability = $\beta(G)$

~~abcdef + abcdefg + abcdeg + abcdfg + abcfg + abcg~~ +

K - chromatic graph of n vertices

$$\beta(G) \geq \frac{n}{K}$$

$$\phi = \sum ab = 0$$

ab: an edge $a \to b$

$$\bar{\phi} = 1 = \prod(\bar{a}+\bar{b}) = \underbrace{f_1 + f_2 + \dots + f_n}_{\text{sop}} = 1$$

Here, $\phi = ab + bc + bd + ce + be + de + ef + eg + fg = 0.$

$$\phi = 1 = (\bar{a}+\bar{b})(\bar{b}+\bar{c})(\bar{b}+\bar{d})(\bar{c}+\bar{e})(\bar{b}+\bar{e})(\bar{d}+\bar{e})(\bar{e}+\bar{f})(\bar{e}+\bar{g})(\bar{f}+\bar{g})$$

$$= (\bar{a}\bar{b}+\bar{b}+\bar{b}\bar{c})(\bar{b}\bar{c}+\bar{d}\bar{c}+\bar{d}\bar{e}+\bar{b}\bar{e})(\bar{b}\bar{d}+\bar{e}+\bar{d}\bar{e}+\bar{b}\bar{e})(\bar{e}+\bar{e}\bar{g}+\bar{e}\bar{f}+\bar{f}\bar{g})(\bar{f}+\bar{g}).$$

$$\cancel{(\bar{a}\bar{b}+\bar{b}+\bar{b}\bar{c})(\bar{b}\bar{c}\bar{d}+\bar{d}\bar{e}+\bar{b}\bar{e}+\bar{b}\bar{c}\bar{e}+\bar{b}\bar{c}\bar{d}\bar{e}+\bar{b}\bar{c}\bar{e}+\bar{b}}$$

$$= (\bar{a}\bar{b}+\bar{b}+\bar{b}\bar{c})(\bar{b}\bar{c}\bar{d}+\bar{b}\bar{c}\bar{e}+\bar{b}\bar{c}\bar{d}\bar{e}+\bar{b}\bar{c}\bar{d}+\bar{c}\bar{d}\bar{e}+\bar{b}\bar{d}\bar{e}+\bar{d}\bar{e}+\bar{b}\bar{e})(\bar{e}\bar{f}+\bar{e}\bar{f}\bar{g}+\bar{f}\bar{g}+\bar{e}\bar{g}).$$

$$= (\bar{a}\bar{b}\bar{c}\bar{d}+\bar{a}\bar{b}\bar{c}\bar{e}+\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}+\bar{a}\bar{b}\bar{d}\bar{e}+\bar{a}\bar{b}\bar{e}+\bar{b}\bar{c}\bar{d}+\bar{b}\bar{c}\bar{e}+\bar{b}\bar{c}\bar{d}\bar{e}+\bar{b}\bar{c}\bar{d}+\bar{b}\bar{d}\bar{e}+\bar{b}\bar{e})(\dots\dots)$$

finally, $\bar{b}\bar{e}\bar{f} + \bar{b}\bar{e}\bar{g} + \bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f} +$

$\bar{b}\bar{c}\bar{d}\bar{f}\bar{g} + \bar{a}\bar{c}\bar{d}\bar{e}\bar{g}$

$\{a, c, d, g\}$ $\{a, c, d, f\}$ $\{b, g\}$

$\{a, e\}$ $\{b, f\}$

---

* If there are 'n' ways to divide V into independent sets such that union of those independent sets produce V, (n ways to divide V into mutually exhaustive sets of independent sets), the graph is said to be n-chromatic, and not uniquely colorable.

* <u>Dominating</u>:- $S \subseteq V$,
every $v \in (V - S)$ $(V \setminus S)$ they are adjacent to any other vertex $s \in S$.

* <u>Minimal dominating set</u>:-
S such that further removal of any vertex from S violates the dominance property.
$S = V$ trivially and proceed removing from there.

* Every maximal I.S is also a Dominating set but the converse is not true.

Minimum size of Minimal dominating set

$\Rightarrow \alpha(G) \Rightarrow$ coefficient of external stability or <u>Domination number</u>

Maximum size of maximal independent set

$\Rightarrow \beta(G) \Rightarrow$ coefficient of internal stability or <u>Independence number</u>.

$$\alpha(G) \leq \beta(G)$$

* <u>Independent set and Dominating set</u> :-

$$\theta = \prod_{\forall V_i \in G} \left(V_i + V_{i_1} + V_{i_2} + V_{i_3} + \dots + V_{i_d} \right)$$

d is degree of V_i ; $V_{i_1}, V_{i_2}, \dots, V_{i_d}$ are adjacent vertices to V_i.

$$\theta = (a+b)(a+b+c+d+e)(b+c+e)(b+d+e)(b+c+d+e+f+g)(e+f+g)(e+f+g).$$

$$= (a + \cancel{ab} + \cancel{ac} + \cancel{ad} + \cancel{ae} + b + \cancel{bc} + \cancel{bd} + \cancel{be})$$
$$(b + \cancel{bd} + \cancel{be} + cd + \cancel{ce} + \cancel{bc} + e)(b+c+d+e+f+g)$$
$$(e + \cancel{ef} + \cancel{eg} + f + \cancel{fg} + g)$$

$$= (a+b)(b+cd+e)(b+c+d+e+f+g)(e+f+g)$$

$$= (ab + acd + ae + b)(\cancel{be} + \cancel{bf} + \cancel{bg} + \cancel{ce} + \cancel{cf} + \cancel{cg} + \cancel{de} + \cancel{df} + \cancel{dg} + e + f + g)$$

$= (ab + acd + ae + b)(e + f + g)$

$= \cancel{abe} + abf + abg + \cancel{acde} + acdf + acdg$
$+ ae + \cancel{aef} + \cancel{aeg} + be + bf + bg$

$= abf + abg + acdf + acdg + ae + be +$
$bf + bg$

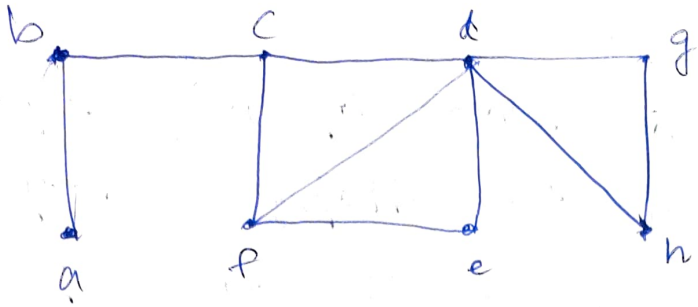
* Interconversion between NP complete problems.
Dominating set $\Leftrightarrow$ Boolean satisfyability.

---

* <u>Hall's marriage theorem</u>:-

A complete matching from V_1 to V_2 in a bipartite graph exists, iff every subset of 'r' vertices in V_1 is collectively adjacent to at least 'r' vertices in V_2 for all values of 'r'.

Th: A covering 'g' of a graph 'G' is minimal iff it doesnot contain any path of length three or more.



covering vertices using edges = $\{ab, cd, ef, gh\}$

$\{ab, cd\}$

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* <u>Graph problems</u> :-

- vertex cover
- clique
- TSP

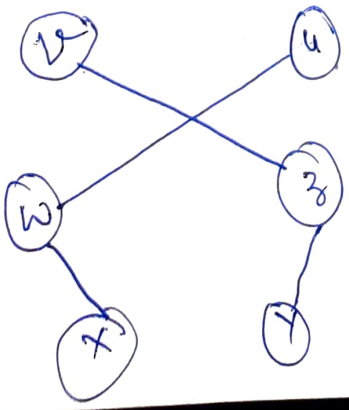
] Approximation Algorithm

Vertex cover → Min. vertex cover $\{\gamma(G)$

Independent set → No two of them are adjacent. (They dont share any edge)

↳ $\beta(G)$

$$\boxed{n = \gamma(G) + \beta(G)}$$



maximal I.S = $\{u, v, x, y\}$

Vertex cover = $\{w, z\}$

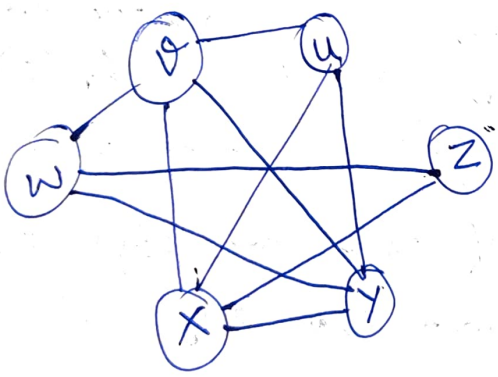
* Vertex cover $\overset{P}{\leq}$ I.S

Vertex cover can be solved by using algorithm for solving I.S, and then applying a Polynomial (P) time complexity process. The above expression represents that relation.

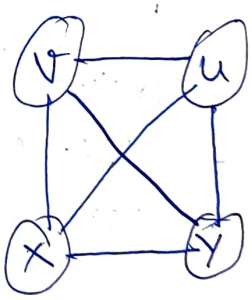
---

* Clique :-

Identify whether a subgraph of the graph is complete. A set of vertices such that all the vertices are connected to each other.



Clique =



Vertex cover of $G(V, E)$:-

(1) $G \rightarrow G'(V, E')$, $e' \in E'$ iff $e' \notin E$ $\forall e'$

G' is a graph where every possible edge except the edge of G are present.

P.T.O

(2) Find clique of $G' \Rightarrow S'$

(3) Take the complement set of vertices of G : $S = V - S'$ //

$$\boxed{|\text{Maximal Matching set}| \leq |\text{min. Vertex Cover set}|}$$

$|X|$ = cardinality of set X.

Matching set algorithm:-

st-1 : $E' = E$, $M = \phi$, $C = \phi$

st-2 : while $E' \neq \phi$

choose a random edge $\{u,v\}$ from E'

$M = M \cup \{u,v\}$

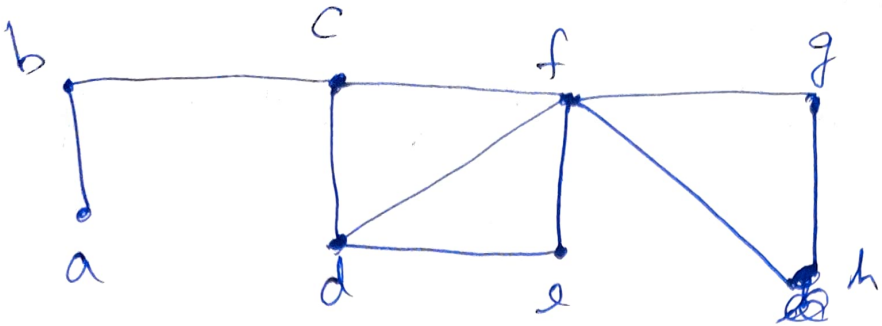
$E' = E' - \{u,v\}$ $\{uv\}$

$- \{$all vertices adjacent to $\{u,v\}\}$

$C = C \cup \{u,v\}$

$$\text{Approximation ratio} = \frac{\text{Actual soln of Approx algo}}{\text{Optimal soln}} \leq \frac{2|M|}{|M|} \leq \underline{\underline{2}}$$

P.T.O



* <u>Apply the algo:-</u>

Max. Matching set = $\{\{a,b\}, \{d,f\}, \{g,h\}\}$

(pick a random edge, remove that edge and all its adjacent edges).

Min. vertex Cover set = $\{a, b, d, f, g, h\}$ size = 6

actual min. vertex cover = $\{b, d, f, g\}$ size = 4

$\frac{6}{4} \leq 2$ //

Travelling salesperson problem :-

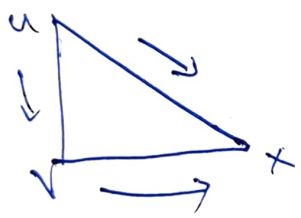
[Complete graph]

No. of Hamiltonian circuits in a complete graph $= \frac{1}{2}(n-1)!$

Finding most optimal among these takes exponential time.

∴ Approximate soln :-

In real life problems, we have :



$\text{Cost}(u \xrightarrow[v]{} x) \geq \text{cost}(u \to x)$

▲ Triangular inequality

∴ (1) Use Prim's / Kruskal's to find the minimum spanning tree for G.

(2) Pre order traversal of T following the edges of T as far as possible whenever it has to backtrack and continue, we don't backtrack, instead we take the direct edge from the present vertex to its preorder successor (possible since it is complete graph and using triangular inequality)